

MATH 566 COMBINATORIAL THEORY LECTURE NOTES

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ABSTRACT. Notes for Math 566 Combinatorial Theory taught by Shiyue Li, during Winter 2026 at the University of Michigan, Ann Arbor. A substantial amount of material is based on Professor Richard Stanley's book titled "Algebraic Combinatorics: Walks, Trees, Tableaux, and More." The course structure and this particular set of material is developed and perfected through conversations with Professor Sergey Fomin and George Seelinger. The author (Shiyue) is fully responsible for any errors and typos, and please contact the author for corrections.

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1. APPLICATION OF LINEAR ALGEBRA TO GRAPH THEORY

In this course, we will encounter many objects that occur everywhere in mathematics, and study their properties with algebraic and combinatorial methods in this course.

1.1. Characteristic polynomial, trace, and eigenvalues. Let $\mathbf{A} = (a_{ij})$ be a $p \times p$ matrix with entries in $\mathbb{C}$. The eigenvalues $\lambda_1, \dots, \lambda_p$ of $\mathbf{A}$ are determined by the *characteristic polynomial*

$$\chi_{\mathbf{A}}(\lambda) := \det(t\mathbf{I} - \mathbf{A}) = \prod_{i=1}^p (t - \lambda_i).$$

The *trace* of $\mathbf{A}$ is the sum of its diagonal entries, which is equal to the sum of its eigenvalues

$$(1) \quad \operatorname{tr}(\mathbf{A}) := \sum_{i=1}^p a_{ii} = \sum_{i=1}^p \lambda_i.$$

Lemma 1.1 Suppose the eigenvalues of a matrix $\mathbf{A}$ are $\mu_1, \dots, \mu_p$.

- (i) Then the eigenvalues of $\mathbf{A} + c\mathbf{I}$ (where c is a scalar) are $\mu_1 + c, \dots, \mu_p + c$.
If the eigenvalues of a matrix $\mathbf{A}$ are $\mu_1, \dots, \mu_p$.

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(ii) Then the eigenvalues of the matrix $\mathbf{A}^\ell$ are $\lambda_1^\ell, \dots, \lambda_p^\ell$.

Lemma 1.2 Let $f(x) \in \mathbb{C}[t]$. Then the polynomial $f(M)$ has eigen values $\{f(\lambda_i)\}$.

Proof. Use Jordan canonical form. $\square$

In this course, most matrices we encounter are real and symmetric (a special case of Hermitian matrices), and so the spectral theorem for Hermitian matrices implies that they are diagonalizable and have real eigenvalues.

1.2. Graph eigenvalues and walks on a graph.

Definition 1.3 A *multiset* M on a set S is an unordered collection of elements in S , if for every $x \in M$, $x \in S$, and the number of times $\mu_M(x)$ that $x \in S$ appear in M is at least 0. The number of times $\mu_M(x)$ that $x \in S$ appear in M is called the *multiplicity* of x in M .

If $\mu_M(x) = 0$ or 1 for all $x \in S$, then M is a set. Two multisets M and M' on the S are said to be equal, if for every $x \in S$, $\mu_M(x) = \mu_{M'}(x)$.

Notation 1.4 Let S be a finite set of size n . We set

$$\binom{S}{k} := \{k\text{-element subsets of } S\}.$$

Note that $\left| \binom{S}{k} \right| = \binom{|S|}{k}$, where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. We set

$$\left(\binom{S}{k} \right) := \{k\text{-element multisets on } S\},$$

called “ n multichoose k .”

Proposition 1.5 For any $n \geq k \geq 0$,

$$\left(\binom{n}{k} \right) = \binom{n+k-1}{k}.$$

Definition 1.6 A *finite graph* G is a triple $G = (V, E, \varphi)$ with

- (i) (Labeled vertices) $V = \{v_1, \dots, v_p\}$,
- (ii) (Labeled edges) $E = \{e_1, \dots, e_q\}$, and
- (iii) (Allowing multiedges and loops) a function $\varphi: E \rightarrow \binom{V}{2}$.

A finite graph is *simple* if the function $\varphi: E \rightarrow \binom{V}{2}$ is injective.

Definition 1.7 Given a graph G , the *adjacency matrix* of G is defined by

$$\mathbf{A}(G) := (|\varphi^{-1}(v_i, v_j)|).$$

Observation 1.8 For any integer $\ell \geq 1$, the (i, j) -entry of the matrix $\mathbf{A}(G)^\ell$ is equal to the number of walks of length ℓ from v_i to v_j in G .

Definition 1.9 Let G be a graph. A *walk* in G of length ℓ from vertex u to vertex v is a sequence $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$ with

- (i) Each v_i is a vertex of G .
- (ii) Each e_j is an edge of G .

- (iii) The vertices of e_i are v_i and v_{i+1} , for $1 \leq i \leq \ell$.
- (iv) $v_1 = u$ and $v_{\ell+1} = u$.

A *closed walk* is a walk such that $v_1 = v_{\ell+1}$.

Theorem 1.10 *Let G be a graph with adjacency matrix $\mathbf{A}(G)$ possessing eigenvalues $\lambda_1, \dots, \lambda_p$. The number of closed walks in G of length ℓ is equal to*

$$f_G(\ell) = \sum_{i=1}^p \lambda_i^\ell.$$

Proof. The number of closed walks in G of length ℓ with start and end at v_i is given by

$$f_G(\ell) = \sum_{i=1}^p (\mathbf{A}(G)^\ell)_{ii} = \text{tr}(\mathbf{A}(G)^\ell) = \sum_{i=1}^p \lambda_i^\ell.$$

□

The following corollary demonstrates how we could determine $f_G(\ell)$ from graph eigenvalues.

Corollary 1.11 *There are $(p-1)^\ell + (-1)^\ell(p-1)$ closed walks of length ℓ in K_p .*

Proof. Consider the complete graph K_p on p vertices. Let $\mathbf{1}$ denote the $p \times p$ matrix with all 1 entries, and let $\mathbf{I}$ denote the $p \times p$ identity matrix. The adjacency matrix of K_p is $\mathbf{A}(K_p) = \mathbf{1} - \mathbf{I}$.

Since the $\text{tr}(\mathbf{1}) = p$ and $\text{rk}(\mathbf{1}) = 1$, the eigenvalues of $\mathbf{1}$ are $p, 0, \dots, 0$.

Since the $\text{tr}(\mathbf{I}) = p$ and $\text{rk}(\mathbf{I}) = p$, the eigenvalues of $\mathbf{I}$ are $1, \dots, 1$.

By Theorem 1.1 (i), the eigenvalues of $\mathbf{A}(K_p)$ is $p-1, -1, \dots, -1$. Therefore, the statement follows from Theorem 1.10. □

Corollary 1.12 *There are $\frac{1}{p}((p-1)^\ell - (-1)^\ell)$ nonclosed walks of length ℓ in K_p from vertex v_i to v_j for $i \neq j$.*

Proof. Let $\mathbf{1}$ denote the $p \times p$ matrix with all 1 entries, and let $\mathbf{I}$ denote the $p \times p$ identity matrix. The adjacency matrix of K_p is $\mathbf{A}(K_p) = \mathbf{1} - \mathbf{I}$.

Since $\mathbf{1}$ and $\mathbf{I}$ commute, by the binomial theorem, we compute

$$(\mathbf{1} - \mathbf{I})^\ell = \sum_{i=1}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} \mathbf{1}^i.$$

Since

$$\mathbf{1}^i = \begin{cases} \mathbf{I}, & \text{if } i = 0 \\ p^{i-1} \mathbf{1}, & \text{if } i > 0. \end{cases}$$

Then we have that

$$\begin{aligned}
(\mathbf{1} - \mathbf{I})^\ell &= \left(\sum_{i=1}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} p^{i-1} \right) \mathbf{1} + (-1)^\ell \mathbf{I} \\
&= \frac{1}{p} \left(\sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} p^i - (-1)^\ell \right) \mathbf{1} + (-1)^\ell \mathbf{I} \\
&= \frac{1}{p} ((p-1)^\ell - (-1)^\ell) \mathbf{1} + (-1)^\ell \mathbf{I}.
\end{aligned}$$

Taking the (i, j) -entry on each side for $i \neq j$, we have that

$$(\mathbf{A}(K_p))_{ij}^\ell = \frac{1}{p} ((p-1)^\ell - (-1)^\ell).$$

□

Proposition 1.13 *Let $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ be nonzero complex numbers such that for every positive integer ℓ ,*

$$\sum_{i=1}^r \alpha_i^\ell = \sum_{j=1}^s \beta_j^\ell.$$

Then $r = s$, and $\{\alpha_i\}_{i=1}^r = \{\beta_j\}_{j=1}^s$, that is, the α_i 's are a permutations of β_j 's.

Corollary 1.14 *The eigenvalues are uniquely determined by the traces of the matrices $\mathbf{M}^\ell$ as ℓ ranges over all positive integers.*

1.3. Inner product, orthogonal basis, Radon transform.

Definition 1.15 An *inner product space* is a vector space V over a field F together with an *inner product*, which is a map

$$\langle \cdot, \cdot \rangle: V \times V \rightarrow F$$

satisfying the following conditions for all vectors $x, y, z \in V$, and scalars $a, b \in F$:

(1) (Conjugate symmetry)

$$\langle x, y \rangle = \overline{\langle y, x \rangle}.$$

(2) (Linearity in the first argument)

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle.$$

(3) (Positivity) If $x \neq 0$, then

$$\langle x, x \rangle > 0.$$

Definition 1.16 For an inner product space V , two vectors $x, y \in V$ are *orthogonal* if $\langle x, y \rangle = 0$. A list of vectors $x_1, \dots, x_m \in \mathbb{C}^n$ is said to be *orthogonal* if $\langle x_i, x_j \rangle = 0$ for all distinct $i, j \in \{1, \dots, m\}$.

Lemma 1.17 *An orthogonal list of nonzero vectors is linearly independent.*

In $\mathbb{R}^2$ and $\mathbb{R}^3$, “orthogonal” has the conventional geometric interpretation of “perpendicular.” The converse can be easily seen to be false.

In this section, we will be using the Kronecker delta function on a set S : for $u, v \in S$, the *Kronecker delta function* is defined as

$$\delta_{uv} := \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Let $\mathbb{Z}_2$ denote the multiplicative group of order 2, with elements $\{0, 1\}$ and the binary operation $+$ modulo 2. For a positive integer $n \geq 1$, let $\mathbb{Z}_2^n$ be the n -fold direct product of $\mathbb{Z}_2$, that is

$$\mathbb{Z}_2^n := \{(a_1, \dots, a_n) \mid a_i \in \mathbb{Z}_2\}.$$

This set satisfies the following properties.

- (i) The set $\mathbb{Z}_2^n$ is endowed with a dot product $\cdot : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$: for any $y, z \in \mathbb{Z}_2^n$,

$$y \cdot z := \sum_{i=1}^n y_i z_i \in \mathbb{Z}_2$$

where the addition on the right hand side is performed modulo $\mathbb{Z}_2$.

- (ii) The arithmetic on this set is as follows: for any $u, v, w \in \mathbb{Z}_2^n$, $u = v + w$ if and only if $v = u + w$ if and only if $w = u + v$.

For an element $u \in \mathbb{Z}_2^n$, we write $|u| = |\{u_i \mid u_i = 1\}|$, called the *Hamming weight*, or *weight* of u .

Let V denote the set of all functions $f: \mathbb{Z}_2^n \rightarrow \mathbb{C}$. This set V has the structure of a vector space with the following properties:

- (1) V is a vector space over $\mathbb{C}$, with the addition operation: for any $f, g \in V$, and any $z \in \mathbb{Z}_2^n$,

$$(f + g)(z) := f(z) + g(z).$$

- (2) The dimension of V over $\mathbb{C}$ is $\dim_{\mathbb{C}} V = 2^n$.¹

- (3) V is an inner product space over $\mathbb{C}$ together with the inner product

$$\langle f, g \rangle = \sum_{u \in \mathbb{Z}_2^n} f(u)g(u).$$

- (4) V has a set of basis $B_1 = \{f_u \mid u \in \mathbb{Z}_2^n\}$ defined as follows. For each $u \in \mathbb{Z}_2^n$, define

$$f_u(v) := \delta_{uv}.$$

This set spans the whole vector space V , because for every $g \in V$, g can be written as a linear combination of elements in B_1 ,

$$g(v) = \sum_{u \in \mathbb{Z}_2^n} g(u)f_u(v) = \sum_{u \in \mathbb{Z}_2^n} g(u)\delta_{uv}.$$

Since B_1 has cardinality 2^n , it is a basis of V .

¹There is an isomorphism $\varphi: V \rightarrow \mathbb{C}^{2^n}$ given by $g \mapsto (g(u_1), g(u_2), \dots, g(u_{2^n}))$.

- (5) V has another basis $B_2 = \{\chi_u \mid u \in \mathbb{Z}_2^n\}$ defined as follows. For each $u \in \mathbb{Z}_2^n$, define

$$\chi_u(v) = (-1)^{u \cdot v}.$$

This set is linearly independent, with the its cardinality the same as the dimension of V . Indeed, for every $u, v \in \mathbb{Z}_2^n$,

$$\langle \chi_u, \chi_v \rangle = \sum_{w \in \mathbb{Z}_2^n} \chi_u(w) \chi_v(w) = \sum_{w \in \mathbb{Z}_2^n} (-1)^{(u+v) \cdot w} = \begin{cases} 2^n & \text{if } u + v = 0, \\ 0 & \text{if } u + v \neq 0. \end{cases}$$

In other words, $\langle \chi_u, \chi_v \rangle \neq 0$ if and only if $u = v$.

Definition 1.18 Given a subset $\Gamma \in \mathbb{Z}_2^n$, and a function $f \in V$, the *discrete Radon transform* (on $\mathbb{Z}_2^n$ with respect to the subset Γ) $\Phi_\Gamma(f)$ is a function in V defined by

$$\Phi_\Gamma(f)(v) := \sum_{w \in \Gamma} f(v + w).$$

For a given subset Γ , the map $\Phi_\Gamma: V \rightarrow V$ is a linear transformation on V .

Theorem 1.19 The eigenvectors of Φ_Γ are the functions $\{\chi_u\}_{u \in \mathbb{Z}_2^n}$. The eigenvalues λ_u corresponding to χ_u is

$$\lambda_u = \sum_{w \in \Gamma} (-1)^{u \cdot w}.$$

Proof. Let $v \in \mathbb{Z}_2^n$. Then we compute

$$\Phi_\Gamma \chi_u(v) = \sum_{w \in \Gamma} \chi_u(v + w) = \sum_{w \in \Gamma} (-1)^{u \cdot (v+w)}.$$

Because there is a factor of $(-1)^{u \cdot v}$ that does not depend on w , we have

$$\Phi_\Gamma \chi_u(v) = \left(\sum_{w \in \Gamma} (-1)^{u \cdot w} \right) (-1)^{u \cdot v} \left(\sum_{w \in \Gamma} (-1)^{u \cdot w} \right) \chi_u(v).$$

Therefore,

$$\Phi_\Gamma(\chi_u) = \left(\sum_{w \in \Gamma} (-1)^{u \cdot w} \right) \chi_u.$$

□

1.4. Walks on a cube.

Definition 1.20 The n -cube is a simple graph with the vertex set $\mathbb{Z}_2^n$, and the edge set

$$\{(u, v) \mid u, v \text{ differ in exactly one component}\}.$$

Example 1.21 Consider the 3-cube in Figure 1. We suppose the following ordering of the vertices

$$\{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Then the adjacency matrix of this cube is

$$\mathbf{A}(C_3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

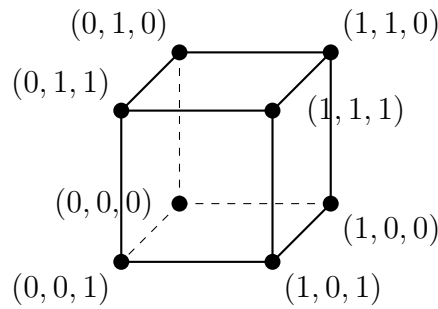


FIGURE 1. The 3-cube.

We will compute the eigenvectors and eigenvalues of the n -cube using Radon transform.

Let Δ be the set

$$\Delta = \{\delta_i \mid \delta_i \text{ is the } i\text{-th unit vector in } \mathbb{Z}_2^n, 1 \leq i \leq n\}.$$

We might think of Δ as the set recording the coordinate of change between two adjacent vertices $u, v \in V(C_n)$. In other words, for $u, v \in \mathbb{Z}_2^n$, $u + v \in \Delta$ if and only if u and v differ in exactly one coordinate. More precisely, $u + v = \delta_i$ if and only if u and v differs at the i th coordinate.

Let $[\Phi_\Delta]$ be the matrix of the linear transformation $\Phi_\Delta: V \rightarrow V$ with respect to the basis B_1 of V .

Lemma 1.22 *The matrix $[\Phi_\Delta] = \mathbf{A}(C_n)$.*

Proof. We aim to show that for any u and v , the (f_u, f_v) -entry of $[\Phi_\Delta]$ is the (u, v) -entry of $\mathbf{A}(C_n)$. Let $z \in \mathbb{Z}_2^n$. We compute

$$\Phi_\Delta f_u(z) = \sum_{w \in \Delta} f_u(z + w) = \sum_{w \in \Delta} f_{u+w}(z)$$

Therefore, we have

$$(2) \quad \Phi_\Delta f_u = \sum_{w \in \Delta} f_{u+w}.$$

For any $v \in \mathbb{Z}_2^n$, Equation (2) implies that

$$[\Phi_\Delta]_{u,v} = \begin{cases} 1 & \text{if } u + w = v, \text{ for some } w \in \Delta \\ 0 & \text{if } u + w \neq v, \text{ for any } w \in \Delta \end{cases}$$

which is equivalent to

$$[\Phi_\Delta]_{u,v} = \begin{cases} 1 & \text{if } u + v \in \Delta, \\ 0 & \text{if } u + v \notin \Delta, \end{cases}$$

which is further equivalent to

$$[\Phi_\Delta]_{u,v} = \begin{cases} 1 & \text{if } (u, v) \in E(C_n), \\ 0 & \text{if } (u, v) \notin E(C_n). \end{cases}$$

□

Corollary 1.23 *The eigenvectors $\{\varepsilon_u\}_{u \in \mathbb{Z}_2^n}$ of $\mathbf{A}(C_n)$ are given by*

$$\varepsilon_u = \sum_{v \in \mathbb{Z}_2^n} (-1)^{u \cdot v} v,$$

with corresponding eigenvalue

$$\lambda_u = n - 2|u|.$$

Moreover, for every $0 \leq i \leq n$, the matrix $\mathbf{A}(C_n)$ has $\binom{n}{k}$ eigenvalues equal to $n - 2k$.

Proof. For any function $g \in V$, by the fact that B_1 is a basis of V , we have

$$g = \sum_v g(v) f_v.$$

From the definition of χ_u , and applying the above equation to χ_u ,

$$\chi_u = \sum_v \chi_u(v) f_v = \sum_v (-1)^{u \cdot v} f_v.$$

The linear transformation Φ_Δ has the same matrix with respect to B_1 as the matrix $\mathbf{A}(C_n)$ with respect to the vertices $V(C_n) = \mathbb{Z}_2^n$, it follows that the eigenvectors of Φ_Δ in terms of B_1 has the same coefficients as the eigenvectors of $\mathbf{A}(C_n)$ in terms of $V(C_n)$.

For each $u \in \mathbb{Z}_2^n$,

$$\lambda_u = \sum_{w \in \Delta} (-1)^{u \cdot w} = \sum_{i=1}^n (-1)^{u \cdot \delta_i} = (n - |u|) - |u| = n - 2|u|.$$

□

Before we only had a formula for the number of closed walks in terms of the eigenvalues of $\mathbf{A}$. What about the non-closed walks?

Definition 1.24 (Unitary matrices) A complex square matrix is *unitary* if its inverse is equal to its conjugate transpose $U^{-1} = U^*$.

Definition 1.25 (Orthogonal matrices) A list of vectors $\{x_1, x_2, \dots\}$ is *orthonormal* if $\langle x_i, x_j \rangle = \delta_{i,j}$. An *orthogonal matrix* is a real square matrix A whose rows and columns are orthonormal vectors. A matrix U is *orthogonal* if and only if $U^T = U^{-1}$.

Theorem 1.26 (Spectral Theorem for Hermitian matrices; [HJ12, Theorem 4.1.5]) A complex square matrix $\mathbf{A}$ is Hermitian if and only if there exists a unitary U and a real diagonal matrix Λ such that $\mathbf{A} = U\Lambda U^*$. Moreover, $\mathbf{A}$ is real symmetric if and only if there is a real orthogonal matrix U and a real diagonal matrix Λ such that $\mathbf{A} = U\Lambda U^{-1}$. In both cases, the matrix U can be chosen to have the orthonormal eigenvectors of $\mathbf{A}$ as columns.

Proposition 1.27 If G is a graph with p vertices with eigenvalues $\lambda_1, \dots, \lambda_p$ as the its eigenvalues, then for $v_i, v_j \in G$, there exists real numbers $c_1, \dots, c_p$ such that for every $\ell \geq 1$,

$$\mathbf{A}(G)_{ij}^\ell = \sum_{i=1}^p c_i \lambda_i^\ell.$$

Moreover, if $U = (u_{r,s})$ is an orthogonal matrix with

$$U^{-1} \mathbf{A}(G) U = \text{diag}(\lambda_1, \dots, \lambda_p),$$

then $c_k = u_{ik} u_{jk}$.

Proof. We prove the stronger statement which requires the matrix U to be orthogonal. Since $\mathbf{A}(G)$ is real and symmetric, there exists a real and orthogonal matrix U , such that $U^{-1} \mathbf{A}(G) U = \text{diag}(\lambda_1, \dots, \lambda_p)$ and $U^{-1} \mathbf{A}^\ell U = \text{diag}(\lambda_1^\ell, \dots, \lambda_p^\ell)$. Therefore,

$$\mathbf{A}^\ell = U \text{diag}(\lambda_1^\ell, \dots, \lambda_p^\ell) U^{-1} = U \text{diag}(\lambda_1^\ell, \dots, \lambda_p^\ell) U^T.$$

Taking the (i, j) -entry yields that

$$(\mathbf{A}^\ell)_{ij} = \sum_k u_{ik} \lambda_k^\ell u_{jk}.$$

□

Corollary 1.28 Let $u, v \in \mathbb{Z}_2^n$, and suppose $|u + v| = k$ i.e., u and v differs in exactly k coordinates. Then the number of walks of length ℓ in C_n between u and v is

$$(\mathbf{A}(C_n)^\ell)_{uv} = \frac{1}{2^n} \sum_{i=0}^n \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{n-k}{i-j} (n-2i)^\ell.$$

In particular, the number of closed walks of length ℓ is

$$(\mathbf{A}(C_n)^\ell)_{uu} = \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (n-2i)^\ell.$$

Proof. By Theorem 1.23, we need to use an orthonormal set of eigenvalues of $\mathbf{A}$, and the norm squares of the eigenvalues of $[\Phi_\Delta]$ is

$$|\varepsilon_u|^2 = \sum_{v \in \mathbb{Z}_2^n} ((-1)^{u \cdot v})^2 = 2^n.$$

Therefore, the vectors $\{\frac{1}{2^{n/2}} E_u\}$ is an orthonormal eigenvectors of $\mathbf{A}$. By Theorem 1.27,

$$(\mathbf{A}(C_n)^\ell)_{uv} = \frac{1}{2^n} \sum_{w \in \mathbb{Z}_2^n} E_{uw} E_{vw} \lambda_w^\ell = \frac{1}{2^n} \sum_{w \in \mathbb{Z}_2^n} (-1)^{(u+v) \cdot w} (n - 2|w|)^\ell.$$

Since $(u+v) \cdot w$ counts how many coordinates w have in common with $u+v$ (this can go from 0 to k), and $|w|$ is the total number of 1 in w (which can go from 0 to n), we split this into two sums

$$(\mathbf{A}(C_n)^\ell)_{uv} = \frac{1}{2^n} \sum_{i=0}^n \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{n-k}{i-j} (n-2i)^\ell.$$

□

Example 1.29 When $k = 1$, we have

$$\begin{aligned} \mathbf{A}(C_n)_{uv}^\ell &= \frac{1}{2^n} \sum_{i=0}^n \left[\binom{n-1}{i} - \binom{n-1}{i-1} \right] (n-2i)^\ell \\ &= \frac{1}{2^n} \left[\sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(n-2i)^{\ell+1}}{n-i} - (-n)^\ell \right]. \end{aligned}$$

One may verify that when $n = 3$, if $\ell = 1$, then we have $\mathbf{A}(C_n)_{uv}^\ell = 1$; if $\ell = 2$, then we have $\mathbf{A}(C_n)_{uv}^\ell = 0$.

1.5. Graph laplacian, and spanning trees. Let G be a finite, connected graph.

Definition 1.30 A *spanning subgraph* of a graph G is a graph H with the same vertex set as G , and such that every edge of H is an edge of G . If G has q , edges, then the number of spanning subgraphs of G is equal to 2^q . A *spanning tree* is a spanning subgraph that is a tree.

The number of spanning trees of G

$$\kappa(G) := \text{the number of spanning trees of } G$$

is called the *complexity* of G .

Example 1.31 (1) For the first few cycle graphs C_n for $n \geq 3$, we have

$$\kappa(C_3) = 3, \kappa(C_4) = 4, \kappa(C_5) = 5, \dots$$

(2) For the first few complete graphs K_n , we have

$$\kappa(K_1) = 1, \kappa(K_2) = 1, \kappa(K_3) = 3, \kappa(K_4) = 16, \kappa(K_5) = 125, \dots$$

Definition 1.32 The *Laplacian matrix* of G is the $|V| \times |V|$ matrix given by

$$\mathbf{L}(G)_{ij} = \begin{cases} -|\varphi^{-1}(v_i, v_j)| & \text{if } v_i \neq v_j \\ \deg(v_i) & \text{otherwise} \end{cases}.$$

For an element σ of the symmetric group S_p , the *sign* of σ is

$$\text{sign}(\sigma) := \begin{cases} 1 & \text{if } \sigma \text{ can be written as even number of transpositions,} \\ -1 & \text{otherwise.} \end{cases}$$

The *determinant* of a $p \times p$ matrix $\mathbf{A} = (a_{ij})$ is

$$\det(\mathbf{A}) := \sum_{\sigma \in S_p} \left(\text{sign}(\sigma) \prod_{i=1}^p a_{i, \sigma(i)} \right)$$

Theorem 1.33 (Laplace expansion formula by minors along a row or a column; [HJ12, 0.3.1]) Let $\mathbf{A}$ be a square matrix in $\mathbb{C}_{n \times n}$. Let $\mathbf{A}_{\widehat{i}, \widehat{j}} \in \mathbb{C}_{n \times n}$ denote the submatrix of $\mathbf{A}$ obtained by deleting row i and column j of $\mathbf{A}$. Then, for any $i, j \in \{1, \dots, n\}$,

$$\det \mathbf{A} = \sum_{k=1}^n (-1)^{i+k} a_{ik} \det \mathbf{A}_{\widehat{i}, \widehat{k}} = \sum_{k=1}^n (-1)^{k+j} a_{kj} \det \mathbf{A}_{\widehat{k}, \widehat{j}}$$

Example 1.34 Let G be the graph as follows. The Laplacian matrix is the 5×5 matrix given by the matrix

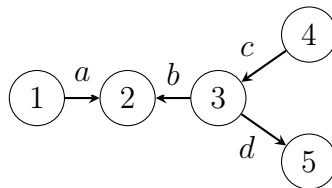
$$\mathbf{L}(G) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Definition 1.35 For $1 \leq r \leq n$ and $1 \leq c \leq m$, the (r, c) -*cofactor* of a square matrix $\mathbf{A}$ is the signed determinant

$$A_{r,c} := (-1)^{r+c} \det(a_{ij})_{i \neq r, j \neq c} = (-1)^{r+c} \det \mathbf{A}_{\widehat{r}, \widehat{c}}.$$

Cofactors appear in the expansion of determinants:

$$\det(\mathbf{A}) = \sum_j a_{rj} A_{r,j} = \sum_i a_{ic} A_{i,c}.$$



The $(1, 1)$ -cofactor at $L_{1,1}$ is 1.

Example 1.36 Let G be the 4 cycle graph C_4 . The Laplacian matrix is

$$\mathbf{L}(C_4) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

The $(4, 4)$ -cofactor $L_{4,4}$ is

$$\det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.$$

Theorem 1.37 (Matrix-Tree theorem) *If G is a finite connected loopless graph with Laplacian $\mathbf{L}$, then the number of spanning trees of G is equal to any cofactor of $\mathbf{L}$.*

1.6. Determinant, Cauchy-Binet theorem, and the Matrix-Tree theorem. For a matrix $X = (x_{ij})$ and a subset S of column indices, we denote

$$X[S] = (x_{ij})_{j \in S}.$$

For nonnegative integers n, m , and subset of $[n + m]$, denote

$$\bar{S} := \{1, \dots, n + m\} \setminus S,$$

and

$$\text{sign}(S) := (-1)^{|\{\bar{s} < s \mid s \in S, \bar{s} \in \bar{S}\}|}.$$

Example 1.38

$$\det(a_{11}) = a_{11}.$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Recall that if $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, then

$$\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}.$$

Theorem 1.39 (Binet–Cauchy theorem) *Let A and B be matrices of size $n \times m$ and $m \times n$ respectively. If $n \leq m$, then*

$$\det(\mathbf{AB}) = \sum_{S \subseteq [m], |S|=n} \det(\mathbf{A}[S]) \det(\mathbf{B}^T[S]).$$

If $n > m$, then $\det(\mathbf{AB}) = 0$.

To prove this, we start with a generalization of the Laplace expansion formula above is the following theorem.

Theorem 1.40 (Laplace expansion theorem; [HJ12, 0.8.9]) *Let $\mathbf{X}$ and $\mathbf{Y}$ be matrices of sizes $n \times (n + m)$ and $m \times (n + m)$, respectively. Then*

$$\det \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \sum_{\substack{S \subset \{1, \dots, n+m\} \\ |S|=n}} \text{sign}(S) \cdot \det \mathbf{X}[S] \cdot \det \mathbf{Y}[\bar{S}]$$

where $\bar{S} = \{1, \dots, n + m\} - S$ and $\text{sign}(S) = (-1)^{\#\{\bar{s} < s \mid s \in S, \bar{s} \in \bar{S}\}}$.

Corollary 1.41 *Let A and B be matrices of sizes $n \times m$ and $m \times n$, respectively, with $n \leq m$. Then*

$$\det \begin{pmatrix} \mathbf{0}_n & \mathbf{A} \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix} = (-1)^n \sum_{|S|=n} \det \mathbf{A}[S] \cdot \det \mathbf{B}^T[S].$$

Proof. Let $\mathbf{X} = (\mathbf{0}_{n \times n} \quad \mathbf{A})$ be the $n \times (n + m)$ matrix, and let $\mathbf{Y} = (\mathbf{B} \quad \mathbf{I}_{m \times m})$ be the $m \times (n + m)$ matrix. The Laplace expansion theorem implies that

$$\begin{aligned} \det \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} &= \sum_{\substack{S \subset \{1, \dots, n+m\} \\ |S|=n}} \text{sign}(S) \cdot \det (\mathbf{0}_{n \times n} \quad \mathbf{A})[S] \cdot \det (\mathbf{B} \quad \mathbf{I}_{m \times m})[\bar{S}] \\ &= \sum_{\substack{S \subset \{1, \dots, n+m\} \\ |S|=n}} \text{sign}(S) \cdot \det \mathbf{A}[S] \cdot \det (\mathbf{B} \quad \mathbf{I}_{m \times m})[\bar{S}] \\ &= \sum_{\substack{S \subset \{1, \dots, n+m\} \\ |S|=n}} \text{sign}(S) \cdot \det \mathbf{A}[S] \cdot \det \mathbf{B}^T[S] \end{aligned}$$

□

Proof of the Binet-Cauchy theorem. By matrix multiplication, we have the following identity:

$$\begin{pmatrix} -\mathbf{I}_n & \mathbf{A} \\ \mathbf{0}_{m \times n} & \mathbf{I}_m \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{A} \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix} = \begin{pmatrix} \mathbf{AB} & \mathbf{0}_n \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix}$$

Take determinants on both sides, we have that

$$\det \begin{pmatrix} -\mathbf{I}_n & \mathbf{A} \\ \mathbf{0}_{m \times n} & \mathbf{I}_m \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{A} \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix} = \det \begin{pmatrix} \mathbf{AB} & \mathbf{0}_n \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix}$$

and therefore,

$$(-1)^{2n} \sum_{|S|=n} \det \mathbf{A}[S] \cdot \det \mathbf{B}^T[S] = \det \mathbf{AB},$$

as desired. □

Definition 1.42 An *orientation* of a graph is an assignment of orders on every edge of G . For an edge e of G such that $\varphi(e) = \{u, v\}$ with $u, v \in V(G)$, if the order is (u, v) , we visualize the orientation on e by drawing an arrow from u to v , u is called the *tail* of e and v is called the *head* of e .

Definition 1.43 The *incidence matrix* of a directed loopless graph is the $|E| \times |V|$ matrix

$$\mathbf{M}(G) := (m_{v,e})_{v \in V(G), e \in E(G)} \text{ with } m_{v,e} = \begin{cases} 1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases}.$$

Example 1.44 Let G be the following graph with its incidence matrix.

$$\mathbf{M}(G) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The following lemma tells us the relationship between the incidence matrix for any orientation of G , and the Laplacian matrix.

Lemma 1.45

$$\mathbf{L}(G) = \mathbf{M}(G)\mathbf{M}(G)^\top.$$

Proof. For any vertices $u, v \in V(G)$,

$$\mathbf{M}(G)\mathbf{M}(G)^\top = (\sum_{e \in E(G)} m_{u,e}m_{v,e})$$

There are 2 cases:

- If $u = v$, then for each $e \in E(G)$,

$$m_{u,e}m_{u,e} = \begin{cases} 1 & \text{if } e \text{ is incident to } v \\ 0 & \text{otherwise} \end{cases}.$$

- If $u \neq v$, then for each $e \in E(G)$,

$$m_{u,e}m_{v,e} = \begin{cases} -1 & \text{if } e \text{ is incident to } u \text{ and } v \\ 0 & \text{otherwise} \end{cases}.$$

The result follows from the definition of the graph Laplacian. $\square$

Lemma 1.46 Let S be a set of $p - 1$ edges of G , and let v be a vertex of G . The determinant of the S -selected submatrix $\mathbf{M}_{\hat{v}}[S]$ of $\mathbf{M}_{\hat{v}}$ is

$$\det(\mathbf{M}_{\hat{v}}[S]) = \begin{cases} \pm 1 & \text{if } S \text{ forms a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}.$$

Proof. If S forms a spanning tree of G , we prove that $\det(\mathbf{M}_{\hat{v}}[S]) = \pm 1$ by induction on the number of vertices p of G . The base case is when $p = 2$. For any S forming a spanning tree of G , $\mathbf{M}_{\hat{v}}(S)$ is the 1×1 matrix (± 1) . We assume, as induction hypothesis that, for any connected graph on $p - 1$ vertices, $v \in V(G)$, and a set of edges S of size $p - 2$ forming a spanning tree of G , $\det \mathbf{M}_{\hat{v}}(S) = \pm 1$. Consider a graph on p vertices, and a set of edges S of size $p - 1$ forming a spanning tree T of G . For any vertex v , there exists an edge e such that the e -column in $\mathbf{M}_{\hat{v}}(S)$ contains exactly one nonzero entry at (u, e) . If $(\mathbf{M}_{\hat{v}}[S])_{\hat{u}, \hat{e}}$ denotes the

matrix $\mathbf{M}_{\hat{v}}[S]$ removing the row and the column containing the nonzero entry, then by cofactor expansion,

$$\det(\mathbf{M}_{\hat{v}}(S)) = \pm \det((\mathbf{M}_{\hat{v}}[S])_{\hat{u}, \hat{e}}).$$

The matrix $(\mathbf{M}_{\hat{v}}[S])_{\hat{u}, \hat{e}}$ is an incidence matrix of a tree on $p - 1$ vertices with one row removed: Let T' denote the tree obtained from T by contracting the edge e into a single vertex w . Then $\mathbf{M}_{\hat{v}}[S]_{\hat{u}, \hat{e}} = \mathbf{M}_{\hat{w}}(T')$. By inductive hypothesis, $\det(\mathbf{M}_{\hat{w}}(T')) = \pm 1$, and so $\det(\mathbf{M}_{\hat{v}}(S)) = \pm \det(\mathbf{M}_{\hat{w}}(T')) = \pm 1$.

If S does not form a spanning tree of G , then there exists a subset $C \subseteq S$ forming a cycle in G . Then $\mathbf{M}_{\hat{v}}(C)$ have linearly dependent columns, which implies that $\det(\mathbf{M}_{\hat{v}}(S)) = 0$. $\square$

The following theorem was first implicitly stated by George Kirchhoff in 1847.

Theorem 1.47 (Matrix-Tree theorem) *If G is a finite connected loopless graph with Laplacian $\mathbf{L}$, then the number of spanning trees of G is equal to any cofactor of $\mathbf{L}$.*

Proof. By Lemma 1.45, the deletion of the r -th row of L and the r -th column of $\mathbf{L}$ satisfies

$$\mathbf{L}_{\hat{r}, \hat{r}} = \mathbf{M}_{\hat{r}} \cdot \mathbf{M}_{\hat{r}}^{\top}.$$

The Binet-Cauchy theorem Theorem 1.39 implies that

$$\det \mathbf{L}_{\hat{v}, \hat{v}} = \sum_{S \subseteq [E], |S|=p-1} (\det \mathbf{M}_{\hat{v}}[S]) (\det \mathbf{M}_{\hat{v}}[S]) = \sum_{S \subseteq [E], |S|=p-1} (\det \mathbf{M}_{\hat{v}}[S])^2.$$

The result follows from Lemma 1.46. $\square$

Lemma 1.48 *If $\mathbf{A} = (a_{ij})$ is a square matrix whose row sums are all 0, then all the cofactors associated with any given row are equal: for every $1 \leq r \leq n$,*

$$A_{r,1} = A_{r,2} = \cdots = A_{r,n},$$

and for every column $1 \leq c \leq m$,

$$A_{1,c} = A_{2,c} = \cdots = A_{n,c}.$$

In other words, all the cofactors are equal.

Proof. Let $\mathbf{A}'$ be the matrix with $a'_{r,1} = a_{r,1} + 1$, and all the other entries the same as those of $\mathbf{A}$. For any $1 \leq k \leq n$, let $\mathbf{A}''$ be the matrix with k -th column equal to the sum of all the columns of $\mathbf{A}'$ and all the other columns the same as $\mathbf{A}'$. Since the construction from $\mathbf{A}''$ using $\mathbf{A}'$ only involves elementary column operations, which preserve the determinant, we obtain

$$\det \mathbf{A}' = \det \mathbf{A}''.$$

Expanding along the k -th column (which consists of exactly one nonzero entry, $a''_{r,k} = 1$), we obtain

$$\det \mathbf{A}'' = A''_{r,k} = A_{r,k}.$$

Since $\det \mathbf{A}' = A_{r,k}$, the (r, k) -cofactor actually does not depend on k , and so

$$A_{r,1} = A_{r,2} = \cdots = A_{r,n}.$$

Similar arguments work for the cofactors along any column. Therefore, all cofactors are equal. $\square$

Proposition 1.49 *Let $\mathbf{L}$ be an $n \times n$ matrix with zero row and column sums, with eigenvalues $\lambda_1, \dots, \lambda_{n-1}, \lambda_n = 0$. Then any cofactor of $\mathbf{L}$ is equal to*

$$\frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

Proof. Since $\mathbf{L}$ is a square matrix with zero row and column sums, all cofactors of $\mathbf{L}$ are equal. We compute the characteristic polynomial of $\mathbf{L}$:

$$\begin{aligned} \det(\mathbf{L} - t \cdot \mathbf{I}_n) &= -t(L_{1,1} + \cdots + L_{n,n}) + \text{some higher order terms} \\ &= -t n L_{1,1} + \text{some higher degree terms} \end{aligned}$$

On the other hand,

$$\det(\mathbf{L} - t \cdot \mathbf{I}_n) = \prod_i (\lambda_i - t) = -t \cdot \lambda_1 \cdots \lambda_{n-1} + \text{higher degree terms}.$$

where the last equality is given by the fact that all cofactors are equal. $\square$

Definition 1.50 A graph G is d -regular if all the vertices have the same degree d .

Proposition 1.51 *If G is a regular graph of degree d , then*

$$\mathbf{L}(G) = d \cdot \mathbf{I} - \mathbf{A}(G).$$

In particular, if G has graph eigenvalues $\lambda_1, \dots, \lambda_p$ with $\lambda_p = d$, then $\mathbf{L}(G)$ has eigenvalues $d - \lambda_1, \dots, d - \lambda_{p-1}, 0$, and moreover,

$$\kappa(G) = \frac{1}{p} (d - \lambda_1) \cdots (d - \lambda_{p-1}).$$

Proof. The first statement follows from the definition. The second statement follows from Lemma 1.2. $\square$

Example 1.52 . Let $G = C_d$ be the d -cube. Then the number of vertices is $|V| = 2^d$. Recall that for each $0 \leq i \leq d$, the adjacency matrix $A(C_d)$ has eigenvalue $d - 2i$ with multiplicity $\binom{d}{i}$. Therefore, the eigenvalues of $L(C_d)$ are $2i$ with multiplicity $\binom{d}{i}$ for $0 \leq i \leq d$. The number of spanning trees in the d -cube is

$$\kappa(C_d) = \frac{1}{2^d} \prod_{i=1}^d (2i)^{\binom{d}{i}}$$

When $d = 3$, $\kappa(C_3) = 384$.

Example 1.53 Let G be the complete graph on p vertices. Since K_p is a d -regular graph with $d = p - 1$, and by Theorem 1.1, the eigenvalues of K_p are

$$p - 1, -1, \dots, -1$$

where the eigenvalue -1 has multiplicity $p - 1$. Therefore, the eigenvalues of $\mathbf{L}(K_p)$ is

$$0, p, \dots, p$$

where the eigenvalue p appears $p - 1$ times. Therefore, the number of spanning trees is

$$\kappa(K_p) = L_{1,1} = \frac{1}{p}(p)^{p-1} = p^{p-2}.$$

1.7. Eulerian digraphs, oriented trees, directed Matrix-Tree theorem.

The following question (and its answer) is considered the beginning of graph theory.

Question 1.54 (Seven Bridges of Königsberg) Can you walk through the city of Königsberg via crossing every one of the seven bridges once and exactly once?

Another question with many use in modern applied mathematics and computer science is related to the de Bruijn sequence.

Definition 1.55 Given a positive number k , a *binary de Bruijn sequence* of degree k is a binary sequence (i.e., consisting of 0, 1) that is periodic with period 2^k and has its list of contiguous subsequences of length k exactly all the binary strengths of length k .

Example 1.56 When $n = 2$, the following sequence is a de Bruijn sequence

$$1, 1, 0, 0, 1, \dots \text{ (repeating).}$$

The contiguous subsequences are 11, 10, 00, 01

Example 1.57 When $n = 3$, the following sequence is a de Bruijn sequence

$$0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0 \dots \text{ (repeating).}$$

The contiguous subsequences are 000, 001, 011, 111, 110, 101, 010, 100.

The following is a generalization of the Leetcode problem <https://leetcode.com/problems/cracking-the-safe/>.

Question 1.58 Generate all de Bruijn sequence of degree k for a given k .

Definition 1.59 A *directed graph* or a *digraph* D consists of vertex set V and E together with a function $\varphi: E \rightarrow V \times V$. In other words, a directed graph is a graph with an orientation.

Let G be a (directed, or undirected) graph.

Definition 1.60 An *Eulerian tour* in G is a closed walk that visits all vertices, by traversing each edge exactly once. If G possesses such a tour, then G is called *Eulerian*.

Theorem 1.61 (Euler) *An undirected graph G is Eulerian if and only if it is connected and the degree of every vertex is even.*

Given a directed graph D , let $h(v)$, called indegree of v , denote the number of edges with v as heads (i.e., edges that go into v), and let $t(v)$, also called outdegree of v , denote the number of edges with v as tails (i.e., edges that go out of v).

Definition 1.62 A directed graph D is *balanced* if $h(v) = t(v)$ for any vertex v .

Lemma 1.63 *For a directed graph D , the following are equivalent:*

- (1) D is Eulerian.
- (2) D is connected undirectedly, and balanced.
- (3) D is connected (directedly) and balanced.

Proof. The implications (1) $\implies$ (3) $\implies$ (2) follows from definition. We prove (2) $\implies$ (1) by strong induction on the number of edges. Fix a vertex v and keep traveling in the graph without repeating an edge. Since D is balanced, and in particular v has the same indegree and outdegree, we will always be able to come back at v . Call all the edges already visited C because it forms a cycle. $D \setminus C$ is a balanced graph; and each of its connected components is balanced and connected undirectedly, and therefore has an Eulerian tour. By inductive hypothesis, we can stitch all Eulerian tours together using C . $\square$

Example 1.64 The de Bruijn graphs D_k is the directed graph whose vertices are the 2^k binary strings of length k , and whose edges are directed edges of the form

$$(a_1, \dots, a_k) \rightarrow (a_2, \dots, a_{k+1}).$$

A de Bruijn sequence of degree k can be interpreted as an Eulerian tour in the graph D_k . Since the graph D_k is balanced and connected (directedly), it is Eulerian.

Definition 1.65 An *arborescence rooted at a vertex v* is a tree whose edges are directed away from v . A subgraph of D is called a *spanning arborescence* if it is an arborescence containing all vertices in D .

Let $\varepsilon(D)$ denote the number of Eulerian tours starting (or ending), and let $\tau(D, v)$ be the number of spanning arborescences rooted at v in D .

Theorem 1.66 (The de Bruijn, van Aardenne-Ehrenfest, Smith, Tutte) *In an Eulerian directed graph D ,*

$$\varepsilon(D) = \tau(D, v) \prod_{u \in V} (h(u) - 1)!.$$

Fix an edge e incident to v . The number of Eulerian tours ending with e is $\varepsilon(D)$. (These tours must begin at v .) For each such tour and each vertex $u \neq v$, consider the edge of the *first arrival* at u .

Observation 1.67 These edges form a spanning arborescence T rooted at v .

For a given Eulerian tour, let us order the $\text{indeg}(u)$ edges coming into each vertex u in the order in which they are traversed by the tour. The resulting collection of linear orderings of edges has the following properties:

- (1) among the incoming edges at a vertex $u \neq v$, the unique edge that belongs to T is the first one in the ordering;
- (2) at v , the edge e is the last one in the ordering.

Lemma 1.68 *Prove that this construction establishes a bijection between*

- (a) *Eulerian tours ending in e , on one hand, and*

(b) pairs consisting of an arborescence T rooted at v and a collection of linear orderings of edges that satisfy (1)-(2).

Observation 1.69 This lemma immediately implies the formula (*).

Proof. We need to construct a map from the set of pairs in (b) to the set of Eulerian tours in (a) which is inverse to the above construction. Suppose we are given a data as in (b): an arborescence T rooted at v and a collection of linear orderings satisfying (1)-(2). Starting with the edge e , we are going to (re)construct the Eulerian tour by moving in the backwards direction; the orderings (taken in reverse) will guide our way. At some point, we will get stuck—necessarily at the distinguished vertex v . All we need to show is that, at that point, we will have traversed all the edges in D .

Suppose this is not true, i.e., we left out some edge(s). Among the vertices u incident to those edges, let us choose one closest in the tree to the root vertex v . (There may be several such u 's.) Obviously $u \neq v$. Since D is balanced, and we have used equally many incoming and outgoing edges at u , one of the incoming edges was not traversed by the constructed walk. In view of condition (1), we therefore did not traverse the unique edge in T that points at u . Let $u' \xrightarrow{e'} u$ be that edge. Notice that u' is closer in T to the root v , compared to u . This contradicts our choice of u , as one of the edges incident to u' has not been traversed. $\square$

Now to compute the number of Eulerian tours, we only need to compute the number of arborescences rooted at a vertex v . We have the following directed version of Matrix-Tree Theorem.

Let D be a digraph (connected or not) with vertex set $V = \{v_1, \dots, v_p\}$ and with ℓ_i loops at vertex v_i . Define the directed Laplacian of D as the matrix

$$\mathbf{L}(D) = (\ell_{i,j}), \text{ with } \ell_{i,j} = \begin{cases} -|\varphi^{-1}(v_j, v_i)| & \text{if } v_i \neq v_j \\ h(v_i) & \text{if } v_i = v_j. \end{cases}$$

Example 1.70 Let D be the graph on vertices $\{1, 2, 3\}$ with directed edges $(1, 2)$, $(2, 3)$, $(1, 3)$ and $(3, 1)$. The directed Laplacian $\mathbf{L}(D)$ is

$$\mathbf{L}(D) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

We compute that $\tau(D, v_1) = 2$, $\tau(D, v_2) = \tau(D, v_3) = 1$.

Theorem 1.71 The number of arborescences of D rooted at v_i is equal to the (i, i) -cofactor of $\mathbf{L}(D)$.

Corollary 1.72 (Matrix-Tree Theorem) The number of spanning tree of any finite connected graph G is equal to any cofactor of its Laplacian.

Proof. Construct a digraph D_G by setting $V(D_G) = V(G)$, and for each $e \in E(G)$ such that $\varphi_G(e) = \{u, v\}$, let e^+ be an edge of D_G such that $\varphi_{D_G}(e^+) = (u, v)$ and

let e^- be an edge of D_G such that $\varphi_{D_G}(e^-) = (v, u)$. The digraph D_G satisfies the following two properties

- (1) For any v of G , there is a bijection between spanning trees of T of G and arborescences T' of D_G rooted at v .
- (2) The directed Laplacian $\mathbf{L}(D_G)$ is equal to the Laplacian $\mathbf{L}(G)$.

□

Example 1.73 An *delivery route* in G is an Eulerian tour in D_G with a distinguished oriented edge (where the delivery starts). The number of delivery route in a graph G is denoted by $\delta(G)$. By the BEST theorem, we have that

$$\delta(G) = 2 |E(G)| \cdot \varepsilon(D_G) = 2 |E(G)| \cdot \kappa(G) \prod_{u \in V(G)} (\deg_G(u) - 1)!.$$

If $G = K_p$, then

$$\delta(G) = 2 \binom{p}{2} p^{p-2} ((p-2)!)^p.$$

When $p = 4$, there are 3072 delivery routes.

2. APPLICATION OF LINEAR ALGEBRA TO PARTIALLY ORDERED SETS

Question 2.1 What is the largest or least number of sets that satisfy a set-theoretic or combinatorial condition?

Question 2.2 What is the largest number of subsets of $[n]$ such that no two of them intersect?

Definition 2.3 A *poset* P , or *partially ordered set*, is a finite set, also denoted P , together with a binary relation $\leq$, called the *partial order*, satisfying the following axioms:

- (P1) (reflexivity) $x \leq x$ for all $x \in P$
- (P2) (antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$
- (P3) (transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$

Example 2.4 Let P be any collection of sets. If $x, y \in P$, then define $x \leq y$ in P if $x \subseteq y$ as sets. It is easy to see that this definition of $\leq$ makes P into a poset. If P consists of *all* subsets of an n -element set S , then P is called a (finite) *boolean algebra* or *boolean poset* of *rank* n and is denoted by B_S . If $S = \{1, 2, \dots, n\}$, then we denote B_S simply by B_n .

Definition 2.5 We say that y *covers* x , if $x < y$ and no element z satisfies $x < z < y$. We then write $x \triangleleft y$ or $y \triangleright x$.

The *Hasse diagram* of a poset P is a planar drawing, with elements of P drawn as dots. If $x < y$ in P (i.e., $x \leq y$ and $x \neq y$), then y is drawn “above” x (i.e., with a larger vertical coordinate). An edge is drawn between x and y if y *covers* x . By the transitivity property (P3), all the relations of a finite poset are determined by the cover relations.

Definition 2.6 Two posets P and Q are *isomorphic* if there is a bijection (one-to-one and onto function) $\varphi: P \rightarrow Q$ such that $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q .

Definition 2.7 A *chain* C in a poset is a totally ordered subset of P , i.e., if $x, y \in C$ then either $x \leq y$ or $y \leq x$ in P . A finite chain is said to have *length* n if it has $n + 1$ elements. Such a chain thus has the form $x_0 < x_1 < \cdots < x_n$. A chain is *maximal* if it is contained in no larger chain. A chain $y_0 < y_1 < \cdots < y_j$ is said to be *saturated* if each y_{i+1} covers y_i . Such a chain need not be maximal since there can be elements of P less than y_0 or greater than y_j .

Definition 2.8 A finite poset is *graded of rank* n if every maximal chain has length n .

Example 2.9 The boolean algebra B_n is graded of rank n .

Definition 2.10 If P is graded of rank n and $x \in P$, then we say that x has *rank* j , denoted $\text{rk}(x) = j$, if any maximal saturated chain of P with top element x has length j .

Implicit in this definition is the fact that all such maximal saturated chains have the same length, a consequence of the poset being graded.

Definition 2.11 If we let $P_j = \{x \in P : \text{rk}(x) = j\}$, then P is a *disjoint union*

$$P = P_0 \uplus P_1 \uplus \cdots \uplus P_n,$$

and every maximal chain of P has the form $x_0 < x_1 < \cdots < x_n$ where $\text{rk}(x_j) = j$. We call P_i the i -th *level* of P .

Definition 2.12 We write $p_j = \#P_j$, the number of elements of P of rank j .

Example 2.13 If $P = B_n$, then $\text{rk}(x) = |x|$ (the cardinality of x as a set), and

$$p_j = |\{x \subseteq \{1, 2, \dots, n\} : \text{rk}(x) = j\}| = \binom{n}{j}.$$

If a graded poset P of rank n has p_i elements of rank i , then define the *rank-generating function*

$$F(P, t) := \sum_{i=0}^n p_i t^i = \sum_{x \in P} t^{\text{rk}(x)}.$$

Example 2.14 $F(B_n, t) = (1 + t)^n$.

Definition 2.15 A graded poset P of rank n is *rank-symmetric* if $p_i = p_{n-i}$ for $0 \leq i \leq n$ and *rank-unimodal* if

$$p_0 \leq p_1 \leq \cdots \leq p_j \geq p_{j+1} \geq p_{j+2} \geq \cdots \geq p_n$$

for some $0 \leq j \leq n$. If P is both rank-symmetric and rank-unimodal, then

$$p_0 \leq p_1 \leq \cdots \leq p_{m-1} \leq p_m \geq p_{m+1} \geq \cdots \geq p_n, \text{ if } n = 2m,$$

$$p_0 \leq p_1 \leq \cdots \leq p_m = p_{m+1} \geq p_{m+2} \geq \cdots \geq p_n, \text{ if } n = 2m + 1.$$

We also say that the sequence $p_0, p_1, \dots, p_n$ itself or the polynomial $F(t) = p_0 + p_1t + \dots + p_nt^n$ is *symmetric* or *unimodal*, as the case may be.

Example 2.16 The boolean algebra B_n is rank-symmetric and rank-unimodal. Since the sequence

$$\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$$

(the n th row of Pascal's triangle) is symmetric and unimodal. Thus the polynomial $(1+t)^n$ is symmetric and unimodal.

Definition 2.17 An *antichain* in a poset P is a subset A of P for which no two elements are comparable, i.e., we can never have $x, y \in A$ and $x < y$.

By definition, the initial question about the maximum size of a collection of subsets of $[n]$ such that no two intersect is equivalent to finding the maximum size of antichain in B_n .

Example 2.18 In any graded poset P of rank n , elements in P_j form an antichain for $1 \leq j \leq n$.

When $P = B_n$, we have Sperner's Theorem, first proven by Emanuel Sperner in 1927.

Theorem 2.19 *The largest collection of subsets of $[n]$ such that no two intersect is the set*

$$\left(\begin{array}{c} [n] \\ \lfloor n/2 \rfloor \end{array} \right) := \text{subsets of cardinality } \lfloor n/2 \rfloor.$$

Definition 2.20 Let P be a graded poset of rank n . The poset P has *Sperner property*, or is a *Sperner poset* if

$$\max\{|A| \mid A \text{ is an antichain}\} = \max\{|P_i| \mid 0 \leq i \leq n\}.$$

In other words, the size of its largest antichain is equal to its largest Whitney number.

Definition 2.21 An *order-matching* from P_i to P_{i+1} is an injective function

$$\mu: P_i \rightarrow P_{i+1}$$

satisfying $x < \mu(x)$ for all $x \in P_i$. Similarly, an order-matching from P_i to P_{i-1} is a one-to-one function

$$\mu: P_i \rightarrow P_{i-1}$$

satisfying $\mu(x) < x$ for all $x \in P_i$.

Proposition 2.22 *Let P be a graded poset of rank n . Suppose there exist an integer $0 \leq j \leq n$ and order-matchings*

$$(3) \quad P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_{j-1} \rightarrow P_j \leftarrow P_{j+1} \leftarrow P_{j+2} \leftarrow \dots \leftarrow P_n.$$

Then P is rank-unimodal and Sperner.

Proof. (Rank-unimodal) Since order-matchings are one-to-one it is clear that

$$p_0 \leq p_1 \leq \cdots \leq p_{j-1} \leq p_j \geq p_{j+1} \geq p_{j+2} \geq \cdots \geq p_n.$$

Hence P is rank-unimodal.

(Sperner) Define a graph G as follows. The vertices of G are the elements of P . Two vertices x, y are connected by an edge if one of the order-matchings μ in the statement of the proposition satisfies $\mu(x) = y$. Thus G is a subgraph of the Hasse diagram of P . The graph G satisfies the following properties:

- (1) The vertices of each of these paths form a chain in P . Thus we have partitioned the elements of P into disjoint chains.
- (2) Since P is rank-unimodal with biggest level P_j , all of these chains must pass through P_j .

Therefore, the number of chains is exactly p_j . Since any antichain A can intersect each of these chains at most once, the cardinality $|A|$ of A cannot exceed the number of chains, i.e., $|A| \leq p_j$. By definition, P is Sperner. $\square$

Definition 2.23 A linear transformation

$$U: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$$

is a raising operator if for every $x \in P_i$, $U(x)$ is a linear combination of elements $y \in P_{i+1}$ satisfying $x < y$.

The first linear algebraic application is the following.

Lemma 2.24 (1) *If there exists an injective raising operator $U: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$, then there exists an order-matching $\mu: P_i \rightarrow P_{i+1}$.*
 (2) *If there exists a surjective raising operator $U: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$, then there exists an order-matching $\mu: P_{i+1} \rightarrow P_i$.*

Proof. Suppose $U: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$ is an injective raising operator. Let $[U]$ denote the matrix of U with respect to the bases P_i of $\mathbb{R}P_i$ and P_{i+1} of $\mathbb{R}P_{i+1}$. Equivalently, the rows of $[U]$ are indexed by the elements $y_1, \dots, y_{p_{i+1}}$ of P_{i+1} and the columns of $[U]$ are indexed by the elements $x_1, \dots, x_{p_i}$ of P_i . Since U is one-to-one, the rank of $[U]$ is equal to p_i . Since the row rank of a matrix equal to its column rank, $[U]$ has p_i linearly independent rows called S .

Let A be the submatrix of $[U]$ consisting of p_i linearly independent rows. In other words, let $A := (U^\top[S])^\top$. Since the rows of A are linearly independent, we have that the determinant of A is nonzero:

$$\det A = \sum_{\pi \in S_{p_i}} \pm a_{1,\pi(1)} \cdots a_{p_i,\pi(p_i)} \neq 0.$$

This implies that there exists a permutation π , such that

$$\pm a_{1,\pi(1)} \cdots a_{p_i,\pi(p_i)} \neq 0.$$

Since U is an raising operator, this implies that, for every $1 \leq k \leq p_i$, we have

$$y_k \succ x_{\pi(k)}.$$

We define a map $\mu: P_i \rightarrow P_{i+1}$ given by, for every $1 \leq k \leq p_i$,

$$\mu(x_k) = y_{\pi^{-1}(k)},$$

which is an order-matching by construction. $\square$

We now apply this to the Boolean algebra B_n to derive that B_n is Sperner. It is enough to show that the Boolean algebra B_n has an injective and surjective raising operator. We define the following.

- (1) Let $U_n = 0$. For $0 \leq i \leq n-1$, let $U_i: \mathbb{R}(B_n)_i \rightarrow \mathbb{R}(B_n)_{i+1}$ be the raising operator defined by

$$U_i(x) = \sum_{x < y \in (B_n)_{i+1}} y.$$

- (2) Let $D_0 = 0$. For $1 \leq i \leq n$, let $D_i: \mathbb{R}(B_n)_i \rightarrow \mathbb{R}(B_n)_{i-1}$ be the lowering operator defined by

$$D_i(y) = \sum_{y > x \in (B_n)_{i-1}} x.$$

- (3) For $0 \leq i \leq n$, let I_i be the identity matrix on $\mathbb{R}(B_n)_i$.

Lemma 2.25 *For $0 \leq i \leq n$, we have*

$$D_{i+1}U_i - U_{i-1}D_i = (n-2i)I_i.$$

Proof. For any x in $(B_n)_i$, we compute

$$D_{i+1}U_i(x) = D_{i+1} \left(\sum_{|y|=i+1, x \subseteq y} y \right) = \sum_{|y|=i+1, x \subseteq y} \sum_{|z|=i, z \subseteq y} z$$

The coefficient of each $z \in (B_n)_i$ is the number of $y \in (B_n)_{i+1}$ such that $x \subseteq y$ and $z \subseteq y$. This number depend on the relationship between z and x and falls into the following 3 cases depending on the cardinality of the set $x \cap z$:

- (1) If $x = z$, then y can be any element in $(B_n)_{i+1}$ covering x . Since every such element correspond to an element of $[n] \setminus x$, there are $n-i$ such y 's.
- (2) If $|x \cap z| = i-1$, then $y = x \cup z \in (B_n)_{i+1}$ is the unique element satisfying $x \subseteq y, z \subseteq y$.
- (3) If $|x \cap z| < i-1$, then there exists no $y \in (B_n)_{i+1}$ such that $x \subseteq y, z \subseteq y$.

This implies that

$$D_{i+1}U_i(x) = (n-i)x + \sum_{|z|=i, |x \cap z|=i-1} z.$$

For any x in $(B_n)_i$, we compute

$$U_{i-1}D_i(x) = U_{i-1} \left(\sum_{|y|=i-1, y \subseteq x} y \right) = \sum_{|y|=i-1, y \subseteq x} \sum_{|z|=i, y \subseteq z} z$$

The coefficient of each $z \in (B_n)_i$ is the number of $y \in (B_n)_{i-1}$ such that $y \subseteq x$ and $y \subseteq z$. This number depend on the relationship between z and x and falls into the following 3 cases depending on the cardinality of the set $x \cap z$:

- (1) If $x = z$, then y can be any element in $(B_n)_{i-1}$ covered by x . Since every such element correspond to an element of x , there are i such y 's.
- (2) If $|x \cap z| = i - 1$, then $y = x \cap z \in (B_n)_{i-1}$ is the unique element satisfying $y \subseteq x, y \subseteq z$.
- (3) If $|x \cap z| < i - 1$, then there exists no $y \in (B_n)_{i-1}$ such that $y \subseteq x, y \subseteq z$.

This implies that

$$\mathbf{U}_{i-1} \mathbf{D}_i(x) = ix + \sum_{|z|=i, |x \cap z|=i-1} z.$$

Putting it together, we have that

$$\mathbf{D}_{i+1} \mathbf{U}_i(x) - \mathbf{U}_{i-1} \mathbf{D}_i(x) = (n - 2i)x.$$

□

2.1. Positive-semidefinite matrices. Recall that a Hermitian matrix is a complex square matrix that is equal to its conjugate transpose.

Definition 2.26 A Hermitian matrix $\mathbf{A} \in \mathbb{C}_{n \times n}$ is positive definite if

$$\text{for all nonzero } x \in \mathbb{C}^n, x^* \mathbf{A} x > 0.$$

A Hermitian matrix $\mathbf{A} \in \mathbb{C}_{n \times n}$ is positive semidefinite if

$$\text{for all } x \in \mathbb{C}^n, x^* \mathbf{A} x \geq 0.$$

Theorem 2.27 ([HJ12, Theorem 7.2.1]) *A Hermitian matrix is positive definite, if and only if all the eigenvalues are positive. A Hermitian matrix is positive semidefinite, if and only if all the eigenvalues are nonnegative.*

Lemma 2.28 *Let $\mathbf{A}$ be an $n \times m$ matrix. Then $\mathbf{A} \mathbf{A}^*$ is a positive semidefinite Hermitian matrix.*

Proof. First, since $(\mathbf{A} \mathbf{A}^*)^* = \mathbf{A} \mathbf{A}$, the matrix $\mathbf{A} \mathbf{A}$ is Hermitian. Second, since for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* \mathbf{A} \mathbf{A}^* \mathbf{x} = \|\mathbf{A}^* \mathbf{x}\|^2 \geq 0$, it is positive semidefinite. □

Theorem 2.29 *The operator U_i is injective for $i < n/2$, and surjective for $i \geq n/2$.*

Proof. We start with the observation that, for all $1 \leq i \leq n$

$$[D_i] = [U_{i-1}]^\top.$$

Indeed, for $a \in (B_n)_{i-1}$ and $b \in (B_n)_i$, the (a, b) -th entry of $[D_i]$ is equal to

$$[D_i]_{a,b} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}.$$

and the (a, b) -th entry of $[U]_{i-1}^\top$ is equal to

$$[U_{i-1}]_{a,b}^\top = [U_{i-1}]_{b,a} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}.$$

By Lemma 2.28, the matrix $[\mathbf{U}_{i-1}][\mathbf{D}_i]$ is positive semidefinite. By Lemma 2.25,

By Theorem 1.1, the eigenvalues of $\mathbf{D}_{i+1}\mathbf{U}_i$ are obtained from the eigenvalues of $\mathbf{U}_{i-1}\mathbf{D}_i$ by adding $n - 2i$.

Suppose $i < n/2$. Then $n - 2i > 0$. Since $[\mathbf{U}_{i-1}][\mathbf{D}_i]$ is positive semidefinite, $\mathbf{D}_{i+1}\mathbf{U}_i$ is positive definite, by Theorem 2.27. In particular, $\mathbf{D}_{i+1}\mathbf{U}_i$ has no zero eigenvalues, and is invertible. Since $\ker(\mathbf{U}_i) \subseteq \ker(\mathbf{D}_{i+1}\mathbf{U}_i) = 0$, $\mathbf{U}_i$ is injective.

Suppose $i \geq n/2$. Then $2i \geq n$ and $2i + 2 > n$. By Lemma 2.25,

$$\mathbf{U}_i\mathbf{D}_{i+1} = \mathbf{D}_{i+2}\mathbf{U}_{i+1} + (2i + 2 - n)\mathbf{I}_{i+1}.$$

By the same arguments as above, $\mathbf{U}_i\mathbf{D}_{i+1}$ is invertible. Since $\text{Im}(\mathbf{U}_i\mathbf{D}_{i+1}) = \mathbb{R}(\mathbf{B}_n)_i$, and $\text{Im}(\mathbf{U}_i\mathbf{D}_{i+1}) \subseteq \text{Im}(\mathbf{U}_i)$, $\text{Im}(\mathbf{U}_i) = \mathbb{R}(\mathbf{B}_n)_i$, and $\mathbf{U}_i$ is surjective. $\square$

2.2. Application of linear algebra to graph theory, via poset theory.

Conjecture 2.30 (Edge reconstruction conjecture, Herary 1964) A simple graph with more than 3 edges G can be uniquely reconstructed from its number of vertices and the multiset of edge-deleted unlabelled subgraphs.

Theorem 2.31 *Let G be a simple graph with p vertices and $3 < q$ edges. If $q > \frac{1}{2}\binom{p}{2}$, then the edge reconstruction conjecture is true.*

2.3. Group action.

Definition 2.32 (Group) A *group* is a set G together with a binary operation $\cdot : G \times G \rightarrow G$ (often written as multiplication) such that the following axioms hold:

- (1) (Closure) For all $a, b \in G$, we have $a \cdot b \in G$.
- (2) (Associativity) For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- (3) (Identity) There exists an element $e \in G$ (called the identity element) such that for all $a \in G$, we have $e \cdot a = a \cdot e = a$.
- (4) (Inverses) For every $a \in G$, there exists an element $a^{-1} \in G$ (called the inverse of a) such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

We denote the group by $(G, \cdot)$ or simply G when the operation is clear from context.

Example 2.33 (1) The set of permutations on n elements, together with composition. This is called the *n -th symmetric group*, denoted by S_n .

(2) The set of permutations of a set X , together with composition. This is called the *permutation group on X* , denoted by S_X .

(3) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ together with the addition $+$.

(4) The nonzero elements of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, together with the multiplication $\times$.

(5) The finite group $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$ together with the addition $+$ modulo n .

(6) The set of isomorphisms of a poset P forms a group, called $\text{Aut}(P)$ together with composition.

Definition 2.34 A group homomorphism $\varphi: G \rightarrow H$ is a map that respects the group operations: for all $x, y \in G$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.

Definition 2.35 Given a group G and a set X of n elements, the group G *acts on* X if there is a group homomorphism $\varphi: G \rightarrow S_X$.

Remark 2.36 The homomorphism for a group action needs not be injective. A group action that comes from an injective homomorphism is called a *faithful* action.

A group action partitions the elements in X into *orbits*, given by an equivalence relation.

Definition 2.37 Let G acts on X . Two elements $x, y \in X$ are G -equivalent if $\pi(x) = y$ for some $\pi \in G$. Any set of G -equivalent elements is called an *orbit*.

Let G be a subgroup of S_n , and G acts on the Boolean algebra B_n by sending

$$\pi: S = \{s_1, \dots, s_k\} \subseteq [n] \mapsto \pi(S) := \{\pi(s_1), \dots, \pi(s_k)\} \subseteq [n].$$

Definition 2.38 The *quotient poset* B_n/G is defined as follows. The elements are orbits of G on X . If O, O' are two orbits, then $O \leq O'$ if there exists $x \in O, y \in O'$ such that $x \leq y$ in B_n .

Proposition 2.39 The quotient poset B_n/G is graded of rank n and rank-symmetric.

Proof. The lengths of maximal chains are all equal to n . For any orbit O , if $X \in B_n$, then let $\bar{X} = [n] \setminus X$. The set X is an orbit if and only if $\bar{X}$ is an orbit. The operation $\bar{\cdot}$ is a bijection between $(B_n/G)_i$ and $(B_n/G)_{n-i}$. $\square$

Theorem 2.40 Let G be a subgroup of $\mathfrak{S}_n$. Then the quotient poset B_n/G is graded of rank n , rank-symmetric, rank-unimodal, and Sperner.

Proof. (Step 1: Rank Symmetry) Let $P = B_n$ and $Q = B_n/G$. By Theorem 2.39, P is graded of rank n and rank-symmetric. It remains to show that P is rank-unimodal and Sperner.

(Step 2: Embedding of posets) First, we define an embedding of posets. For each orbit $\omega \in Q_i = (B_n)_i/G$, define the *orbit sum*

$$v_\omega = \sum_{x \in \omega} x \in \mathbb{R}P_i.$$

Since distinct orbits are disjoint, the orbit sums have disjoint supports, and are therefore linearly independent. Define the injective linear map

$$\varphi_i: \mathbb{R}Q_i \rightarrow \mathbb{R}P_i, \quad \omega \mapsto v_\omega.$$

(Step 3: Raising operators) Next, we define a family of raising operator on Q , using the raising operators U_i on P_i . For any $\omega \in P_i$, the element $U_i(v_\omega)$ is a linear combination of orbit sums at level $i+1$, because $x < y$ in P if and only if $\pi(x) < \pi(y)$ for every $\pi \in G$. If the image of an element $v_\omega \in \mathbb{R}P_i$ under U_i is written as

$$U_i(v_\omega) = \sum_{\omega' \in Q_{i+1}} c_{\omega, \omega'} v_{\omega'},$$

we can define $\widehat{U}_i: \mathbb{R}Q_i \rightarrow \mathbb{R}Q_{i+1}$ by

$$\widehat{U}_i(\omega) := \sum_{\omega' \in Q_{i+1}} c_{\omega, \omega'} \omega'.$$

We need to show that $\widehat{U}_i$ is a raising operator, i.e., if $c_{\omega, \omega'} \neq 0$, then $\omega < \omega'$ in Q . If $c_{\omega, \omega'} \neq 0$, since $c_{\omega, \omega'}$ is also the coefficient in

$$U_i(v_\omega) = \sum_{\omega' \in Q_{i+1}} c_{\omega, \omega'} v_{\omega'} = \sum_{\omega' \in Q_{i+1}} c_{\omega, \omega'} \sum_{x' \in \omega'} x'$$

there exists some $x' \in \omega'$, and $x \in \omega$ such that $x < x'$ in P . By definition of Q , $\omega < \omega'$ in Q . Furthermore, by construction, the operator $\widehat{U}_i$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}Q_i & \xrightarrow{\varphi_i} & \mathbb{R}P_i \\ \downarrow \widehat{U}_i & & \downarrow U_i \\ \mathbb{R}Q_{i+1} & \xrightarrow{\varphi_{i+1}} & \mathbb{R}P_{i+1} \end{array}.$$

(Step 4: Injectivity of Raising Operators) Finally, we prove that $\widehat{U}_i$ is an injective raising operator. Suppose $i < n/2$ and $\widehat{U}_i(w) = 0$ for some $w \in \mathbb{R}Q_i$. Then

$$U_i(\varphi_i(w)) = \varphi_{i+1}(\widehat{U}_i(w)) = 0.$$

By a previous theorem, U_i is injective on all of $\mathbb{R}P_i$ for $i < n/2$, so $\varphi_i(w) = 0$. Since φ_i is itself injective, $w = 0$. Thus $\widehat{U}_i$ is one-to-one for $i < n/2$.

Surjectivity of $\widehat{U}_i$ for $i \geq n/2$ is analogous. Therefore, $Q = B_n/G$ is rank-unimodal and Sperner. $\square$

Remark 2.41 All the statements apply to the lowering operator D_i and a similarly constructed $\widehat{D}_i$. In particular, D_i is injective for $i > \frac{n}{2}$, and surjective for $i \leq \frac{n}{2}$.

Theorem 2.42 Let G be a simple graph with p vertices and $q > \frac{1}{2} \binom{p}{2}$ edges. Then G is edge-reconstructible.

Proof. (Step 1: Construction of the poset of isomorphism classes of graphs on p vertices) Let P_q be the poset of all simple, labeled graphs with q edges on the vertex set $[p]$, with the partial order $G < G'$ if G is a labeled subgraph of G' . The poset P_q is graded of rank $\binom{p}{2}$, and for a graph with p vertices, $\text{rk}(G) = |E(G)|$.

Let $\mathbb{R}P_q$ denote the real vector space with basis P_q . Define a linear map

$$D_i: \mathbb{R}P_i \rightarrow \mathbb{R}P_{i-1}, G = \sum_{e \in E(G)} G \setminus e.$$

The symmetric group S_p acts on the poset P_q , by permuting the vertices $[p]$. Therefore, this induces a quotient poset

$$Q_q := P_q/S_n := \{\text{isomorphism classes of graphs on } p \text{ vertices}\},$$

with the partial order

$$[G] \leq [G']$$

if there exists a vertex labeling of G and G' such that G is a subgraph of G' .

(Step 2: $ER \iff$ Injectivity of lowering operator) We have the following commutative diagram of linear maps.

$$\begin{array}{ccc} \mathbb{R}Q_i & \xrightarrow{\varphi_i} & \mathbb{R}P_i \\ \downarrow \widehat{D}_i & & \downarrow D_i \\ \mathbb{R}Q_{i-1} & \xrightarrow{\varphi_{i-1}} & \mathbb{R}P_{i-1} \end{array} \quad .$$

By previous theorems, and remark, the operator $\widehat{D}_i$ is injective for $i > \frac{1}{2}\binom{p}{2}$. This implies that for every pair of non-isomorphic graphs G and G' with $i > \frac{1}{2}\binom{p}{2}$ edges (which correspond to two distinct classes $[G] \neq [G']$ in Q_i , their images under $\widehat{D}_i$ are distinct, by injectivity

$$\sum_{e \in E(G)} [G \setminus e] = \widehat{D}_i([G]) \neq \widehat{D}_i([G']) = \sum_{e \in E(G')} [G' \setminus e].$$

This is equivalent to saying that the multiset of 1-edge deleted unlabeled subgraphs of a simple graph G with at least $\frac{1}{2}\binom{p}{2}$ edges uniquely determine the isomorphism class of G . $\square$

2.4. Other ways to unimodality: log-concavity, and real-rootedness.

Definition 2.43 A sequence $a_0, \dots, a_n$ of real numbers is *log-concave* if for all $1 \leq i \leq n-1$. A sequence is *strongly log-concave* if

$$\frac{a_i^2}{\binom{n}{i}^2} \geq \frac{a_{i-1}a_{i+1}}{\binom{n}{i-1}\binom{n}{i+1}}.$$

A sequence of real numbers has *no internal zeros* if for all $i < j < k$, we have the implication

$$a_i \neq 0, a_k \neq 0 \implies a_j \neq 0.$$

Proposition 2.44 Let $\alpha = (a_0, a_1, \dots, a_n)$ be a sequence of non-negative real numbers with no internal zeros. If α is log-concave, then it is unimodal.

Proof. Suppose for contradiction that the sequence $a_0, a_1, \dots, a_n$ is not unimodal. Then there exists an index $1 \leq i \leq n-1$ such that at least one of the two holds:

- either $a_{i-1} > a_i \leq a_{i+1}$ with $a_{i+1} \neq 0$,
- or $a_{i-1} \geq a_i < a_{i+1}$ with $a_{i-1} \neq 0$.

In the first case, if $a_{i+1} = 0$, then $a_i = 0$ and the triple a_{i-1}, a_i, a_{i+1} is not a witness of failure of unimodality, in which case, we can always find another index j such that one of these two conditions hold for j . Similarly for the second case, we will need $a_{i-1} = 0$. In any case, $a_i^2 < a_{i-1}a_{i+1}$, a contradiction. $\square$

Theorem 2.45 (Issac Newton 1707) If a polynomial $P(x) \in \mathbb{R}[x]$ is real-rooted, then its coefficient sequence is strongly log-concave.

Lemma 2.46 (Rolle's Theorem) If f is a real-valued, continuous function on a closed proper interval $[a, b]$ such that f is differentiable on (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Proof. The goal is to reduce the polynomial $P(x)$ to a quadratic, real-rooted polynomial containing coefficients a_{i-1}, a_i, a_{i+1} , such that the discriminant of the quadratic polynomial yields $a_i^2 \geq a_{i-1}a_{i+1}$. The reduction is carried out via differentiation and taking the reciprocal of polynomials.

Let

$$P(x) = \sum_{i=0}^n b_i x^i = \sum_{i=0}^n \binom{n}{i} a_i x^i$$

be a real-rooted polynomial, and denote $\deg P(x) = m \leq n$. By the Fundamental Theorem of Algebra, and the assumption that $P(x)$ is real-rooted, $P(x)$ has exactly m real zeros counting multiplicities.

Claim 1. For any $0 \leq i \leq m$, the i -th derivative of $P(x)$

$$Q(x) := \frac{d^{i-1}}{dx^{i-1}} P(x)$$

is a real-rooted polynomial.

Proof of Claim 1. By induction, it suffices to show that the first derivative of a real-rooted polynomial is real-rooted. We analyze the locations of the roots of $P'(x)$ given the locations of the roots of $P(x)$.

(1) Suppose α is root of $P(x)$ with multiplicity $r > 1$. Then

$$P(x) = (x - \alpha)^r L(x)$$

where $L(x)$ is a polynomial with $L(\alpha) \neq 0$. Then

$$P'(x) = r(x - \alpha)^{r-1} L(x) + (x - \alpha)^{r-1} L'(x),$$

which implies that

$$P'(\alpha) = 0$$

and α is root of $P'(x)$.

(2) Suppose $\alpha < \beta$ are two distinct roots of $P(x)$, then by Rolle's Theorem, there exists a real number γ such that $\alpha < \gamma < \beta$ and $P'(\gamma) = 0$.

This shows that $P'(x)$ has at most $m - 1$ real-roots.

Claim 2. The reciprocal of a real-rooted polynomial is real-rooted.

Proof of Claim 2. By definition, the reciprocal of a real-rooted polynomial $P(x)$ of degree m is

$$R(P(x); x) := x^m P(1/x),$$

where $R(P(x); x)$ is a polynomial of degree at most m . If $P(x)$ is real-rooted, then $R(x)$ has roots exactly the reciprocals of those nonzero roots of $P(x)$.

By explicit computations, the polynomial

$$Q(x) := \frac{d^{m-i-1}}{(dx)^{m-i-1}} \left(R \left(\frac{d^{i-1}}{(dx)^{i-1}} P(x); x \right) \right) = \frac{m!}{2} (a_{i-1}x^2 + 2a_x + a_{i+1})$$

If $a_{i-1} = 0$ or $a_{i+1} = 0$, then it is straightforward that $a_i^2 \geq a_{i-1}a_{i+1}$.

Otherwise, $Q(x)$ is quadratic polynomial with real-roots, and by the quadratic formula, the discriminant

$$\Delta_{Q(x)} = 4(a_i^2 - a_{i-1}a_{i+1}) \geq 0,$$

which implies that $a_i^2 \geq a_{i-1}a_{i+1}$. □

2.5. Young diagram and q -binomial coefficients.

Definition 2.47 Given an integer $n \geq 0$, a *partition* λ of an integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers $\lambda_i \geq 0$ such that

- (1) $\lambda_1 \geq \lambda_2 \geq \dots$ and
- (2) $n = \sum_{i \geq 1} \lambda_i$.

Each λ_i is called a *part* of λ , and we use $\lambda \vdash n$ for a partition λ of n .

Partitions of integers have been widely studied in all areas of mathematics, with number theory, combinatorics, and representation theory prominently featuring partitions.

Theorem 2.48 (Hardy-Ramanujan, 1918) *If $p(n)$ denote the number of partitions of a nonnegative integer n , then*

$$p(n) \sim \frac{1}{n4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \rightarrow \infty$.

2.6. Interlude: Generating functions. Given a sequence of integers $\alpha = (a_0, a_1, \dots)$, the *generating function* of α is the power series (a polynomial if α is finite)

$$G_\alpha(x) := \sum_{i \geq 0} a_i x^i.$$

Leonard Euler (1707-1783) was the first known person to systematically use *generating functions* for counting problems.

Theorem 2.49 (Euler 1740) (1)

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

(2)

$$\sum_{n=0}^{\infty} \#\{\text{partitions of } n \text{ with parts } \leq k\} x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$$

(3)

$$\sum_{n=0}^{\infty} \#\{\text{partitions of } n \text{ with } \leq k \text{ parts}\} x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$$

(4)

$$\sum_{n=0}^{\infty} \#\{\text{partitions of } n \text{ with distinct parts}\} x^n = (1+x)(1+x^2)(1+x^3)\cdots$$

(5)

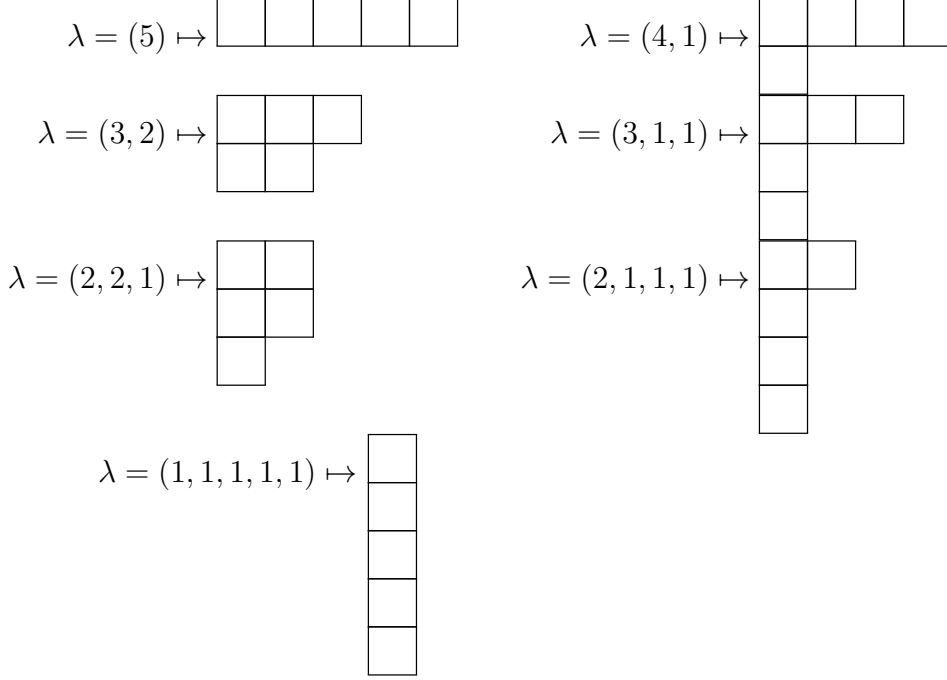
$$\sum_{n=0}^{\infty} \#\{\text{partitions of } n \text{ with odd parts}\} x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

Remark 2.50 The last two formulas are equal to each other

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

2.7. The poset of Young diagrams. A partition can be visualized using an arrangement of left-adjusted boxes, with λ_i boxes in the i th row from the top.

Example 2.51 The Young diagrams that depict all the partitions of the integer 5.



Definition 2.52 (Poset structure on $L(m, n)$) Given positive integers m and n , let $L(m, n)$ be the set of all partitions with at most m parts, and with the largest part at most n . In other words, $L(m, n)$ consist of all partitions whose Young diagrams can fit into a $m \times n$ rectangle. There is a poset structure on $L(m, n)$ given by the partial order: if $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, then $\lambda \leq \mu$ if for every i , $\lambda_i \leq \mu_i$ (appending zero if necessary). In other words, $\lambda \leq \mu$ if the Young diagram of λ is physically covered by that of μ .

The poset $L(m, n)$ is graded of rank mn , given by the rank function

$$\text{rk}(\lambda) = \sum_i \lambda_i = |\lambda|.$$

Proposition 2.53 *The poset $L(m, n)$ is rank-symmetric.*

Proof. We establish a bijection between λ and its complement in R_{mn} . For each λ , let $\bar{\lambda}$ be the complement of λ , i.e., for every $1 \leq i \leq m$,

$$\bar{\lambda} := (\bar{\lambda}_i), \text{ with } \bar{\lambda}_i := n - \lambda_{n-i}.$$

This defines a bijection from $L(m, n)_i$ to $L(m, n)_{mn-i}$. □

We are interested in counting the total number of elements in $L(m, n)$ and the number of elements of $L(m, n)$ of rank i , which are also called “Gaussian coefficients.”

Proposition 2.54 *The total number of elements in $L(m, n)$ is*

$$|L(m, n)| = \binom{m+n}{n}$$

Proof. The set of Young diagrams in a $m \times n$ rectangle is in bijection with the number of lattice paths going from the lower-left corner to the upper-right corner of the same rectangle. The number of lattice paths is given by going right n times in a path of length $m + n$, and so is equal to

$$\binom{m+n}{n}.$$

□

Remark 2.55 The proof above gives another proof of the fact that for any $n \geq k \geq 0$,

$$\binom{\binom{n}{k}}{k} = \binom{n+k-1}{k},$$

by establishing a bijection between k -multisets of $[n]$ and lattice paths from lower-left corner to the upper-right corner of the $(n + k - 1) \times k$ rectangle.

For any positive integer j , the q -analogue of the integer j is defined as

$$[j]_q := \sum_{i=0}^{j-1} q^i = 1 + q + q^2 + \dots + q^{j-1}.$$

The q -factorial, or the q -analogue of the factorial of j is defined as

$$[j]_q! := \prod_{i=1}^j [i]_q = [1]_q [2]_q \cdots [j]_q,$$

with the convention that $[0]_q! = 1$. For $k \geq j \geq 0$, the q -binomial coefficient, or Gaussian coefficient is defined as

$$\begin{bmatrix} k \\ j \end{bmatrix}_q := \frac{[k]_q!}{[j]_q! [k-j]_q!}.$$

You have seen the ingredients of Gaussian coefficients in a previous homework problem. The poset $B_n(q)$ of subspaces of an n -dimensional vector space over $\mathbb{F}_q$ has number of elements of rank k equal to

$$|B_n(q)_k| = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Lemma 2.56 *For any $k \geq 1$, q -binomial coefficients satisfy the recursive identity*

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} k-1 \\ j \end{bmatrix}_q + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q,$$

with the initial condition that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1, \begin{bmatrix} k \\ j \end{bmatrix}_q = 0$$

if $j < 0$, or $j > k$.

Proof. We prove this by a computation. By definition, the right hand side

$$\begin{bmatrix} k-1 \\ j \end{bmatrix}_q + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q = \frac{[k-1]_q!}{[j]_q![k-1-j]_q!} + q^{k-1} \frac{[k-1]_q!}{[j-1]_q![k-j]_q!}.$$

Extracting common factors, the above becomes

$$\frac{[k-1]_q!}{[j-1]_q![k-1-j]_q!} \left(\frac{1}{[j]_q} + \frac{q^{k-j}}{[k-j]_q} \right).$$

Creating a common denominator in the sum, the above becomes

$$\frac{[k-1]_q!}{[j-1]_q![k-1-j]_q!} \cdot \frac{[k-j]_q + q^{k-j}[j]_q}{[j]_q[k-j]_q}.$$

By the definition of the q -analogue of an integer, the numerator is equal to

$$[k-j]_q + q^{k-j}[j]_q = \sum_{i \geq 0}^{k-j-1} q^i + q^{k-j} \sum_{i \geq 0}^{j-1} q^i = \sum_{i \geq 0}^{k-j-1} q^i + \sum_{i \geq 0}^{j-1} q^{k-j+i} = \sum_{i \geq 0}^{k-1} q^i = [k]_q.$$

Therefore, we have

$$\begin{bmatrix} k-1 \\ j \end{bmatrix}_q + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q = \frac{[k-1]_q!}{[j-1]_q![k-1-j]_q!} \cdot \frac{[k]_q}{[j]_q[k-j]_q} = \begin{bmatrix} k \\ j \end{bmatrix}_q$$

□

Remark 2.57 The q -analogues of mathematical objects tend to become familiar objects when $q = 1$.

Theorem 2.58 If we let $p_i(m, n)$ denote the number of elements of $L(m, n)$ of rank i , then

$$\sum_{i \geq 0}^{mn} p_i(m, n) q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

Proof. Let $P(m, n)(q)$ denote the function in q

$$\sum_{i \geq 0}^{mn} p_i(m, n) q^i.$$

We will show that it satisfies the recursive formula of $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ together with the initial conditions.

For the initial conditions: The lattice $L(0, 0)$ is the empty rectangle, which contains only the empty partition $\emptyset$. Therefore,

$$P(0, 0) = 1.$$

The lattice $L(m, n)$ is undefined, if $m < 0$ or $n < 0$. Therefore, the generating function $P(m, n)$ is an empty sum, and

$$P(m, n) = 0.$$

For the recursive identity, the statement

$$P(m, n) = P(m, n-1) + q^n P(m-1, n)$$

is equivalent to the statement that, for each $0 \leq i \leq mn$,

$$p_i(m, n) = p_i(m, n-1) + p_{i-n}(m-1, n).$$

A Young diagram λ which fits in an $m \times n$ rectangle can come from 2 possible constructions:

- (1) If λ does not contain the rightmost box in the first row, then λ fits in a rectangle of size $m \times (n-1)$. There are $p_i(m, n-1)$ such partitions.
- (2) If λ does contain the right most box, then λ contains the whole first row, and so λ can be created uniquely by adding a first row to a partition λ' with $|\lambda| = i - n$, which fits in a rectangle of size $(m-1) \times n$. There are $p_{i-n}(m-1, n)$ such partitions.

□

2.8. Wreath product.

Definition 2.59 The group G_{mn} consists of permutations of boxes in a $m \times n$ rectangle which permute the elements within each row, and then permute the rows.

Theorem 2.60 Let $R = R_{mn}$ be the set of mn boxes in an $m \times n$ rectangle. The quotient poset B_R/G_{mn} is isomorphic to $L(m, n)$.

Proof. (Bijection) Every orbit O of the action of G_{mn} on B_R contains exactly one Young diagram (exactly the one that is left adjusted). There is a well-defined, bijective map

$$\varphi: B_R/G_{mn} \rightarrow L(m, n), D_\lambda \mapsto \lambda,$$

where D_λ is the unique Young diagram of an orbit O .

(Poset structure) For orbits O and O' of B_R/G_{mn} , we claim that $O \leq O'$ if and only if $\varphi(O) = \lambda \leq \lambda' = \varphi(O')$. Suppose $O \leq O'$, then by definition there exists Young diagrams $D \in O$ and $D' \in O'$ such that $D \subseteq D'$. Then by the definition of partitions associated to Young diagrams, we have that for each $1 \leq j \leq m$, $\lambda_j \leq \lambda'_j$. □

Corollary 2.61 The poset $L(m, n)$ is rank-symmetric, rank-unimodal, and Sperner.

2.9. Young tableaux and the hook length formula. We are interested in counting walks in the infinite poset $\mathbf{Y}$ consisting of all partitions of nonnegative integers. A walk from the empty partition to a specified partition λ on rank n is equivalent to a *standard Young tableau*.

Then to achieve a partition in $\mathbf{Y}$ is equivalent to counting the number of standard Young tableaux

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

Definition 2.65 Given a shape $\vdash n$, a composition $\mu = (\mu_1, \dots, \mu_k)$ and a *semistandard Young tableau* (SSYT) of $\vdash$ is a filling of $\vdash$ using content μ such that the followings hold:

- There are many combinatorial questions that we would like to answer in this class regarding/using combinatorial theory of Young tableaux.

- $$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

Definition 2.66 Let u be a square of the Young diagram of the partition λ . Define the hook $H(u)$ of u to be the set of all squares directly to the right of u or directly below u , including u itself. The size of $H(u)$ is called the *hook length* of u .

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where u ranges over all boxes of the Young diagram of λ .

Example 2.68 Consider the example of the partition

$$\lambda = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \lambda' = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

2.10. Counting walks in Young's lattice. We set some notation. Let $w = A_n A_{n-1} \dots A_1$ be some word in U and D where U is going up and D is going down. For $\lambda \vdash n$, write $\alpha(w, \lambda)$ be the number of Hasse walks in $\mathbf{Y}$ of type w , starting from the partition $\emptyset$ and ending at the partition λ .

Question 2.69 What is the relationship between the number of walks $\alpha(w, \lambda)$ and f^λ ?

For specific w , we have some easy answers.

Observation 2.70 The number of Young tableaux $f^\lambda = \alpha(U^n, \lambda)$.

Observation 2.71 The number of walks of type $D^n U^n$ starting at $\emptyset$ and reaching some partition $\lambda \vdash n$, and ending at $\emptyset$ is

$$\alpha(D^n U^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

But it is not clear such a formula exists, or the number is given w and λ is nonzero. We have the following criteria.

Lemma 2.72 Let $w = D^{s_k} U^{r_k} \dots D^{s_1} U^{r_1}$ for $r_i \geq 0$ and $s_i \geq 0$. Let $\lambda \vdash n$. Then there exists a walk w from $\emptyset$ to λ if and only if

$$\sum_{i=1}^k (r_i - s_i) = n \text{ and } \sum_{i=1}^j (r_i - s_i) \geq 0, 1 \leq j \leq k.$$

Proof. We prove the necessity of these two conditions. Suppose there exists a walk of type w from $\emptyset$ to λ . First, since U is up, D is down,

$$\sum_{i=1}^k (r_i - s_i) = \text{the rank of the end} = \text{rk}(\lambda) = n.$$

Second, for every $1 \leq j \leq k$,

$$\sum_i^j (r_i - s_i) = \text{the level after we execute the first } j \text{ pairs of ups and downs} \geq 0.$$

□

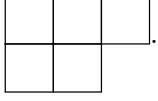
Definition 2.73 For $\lambda \vdash n$ and w a word, if $\alpha(w, \lambda) \geq 1$, then w is called a *valid λ -word*.

We will try to derive a formula for the valid w . We start with the following construction:

Let $\mathbb{R}Y_j$ be the real vector space with basis Y_j . Define linear transformations

$$U_i: \mathbb{R}Y_i \rightarrow \mathbb{R}Y_{i+1}, U_i(\lambda) = \sum_{\mu \vdash i+1, \lambda < \mu} \mu,$$

$$D_i: \mathbb{R}Y_i \rightarrow \mathbb{R}Y_{i-1}, D_i(\lambda) = \sum_{\mu \vdash i-1, \mu < \lambda} \mu.$$

Example 2.74 Let $\lambda =$ .

$$U_5(\text{Young diagram for } \lambda) = \text{Young diagram for } (4,3) + \text{Young diagram for } (4,2,1) + \text{Young diagram for } (3,3,1)$$

and

$$D_5(\text{Young diagram for } \lambda) = \text{Young diagram for } (2,2) + \text{Young diagram for } (3,1).$$

Observation 2.75 If $d(\lambda)$ is the number of distinct parts of λ , then $U_i(\lambda)$ is the sum of $d(\lambda) + 1$ terms, and $D_i(\lambda)$ is the sum of $d(\lambda)$ terms.

Lemma 2.76 For any $i \geq 0$, the operators satisfy the Heisenberg commutation relation

$$D_{i+1}U_i - U_{i-1}D_i = I_i.$$

Remark 2.77 The Heisenberg commutation relation appears in Weyl algebras. Consider the polynomial ring $\mathbb{R}[x]$. Let D, U act on $\mathbb{R}[x]$ by

$$D(f) = f', \quad U(f) = xf.$$

Leibnez rule implies $DU - UD = I$.

Proof. For any $\lambda \vdash i$. We consider the coefficient of a diagram $\mu \vdash i$ in the image of $D_{i+1}U_i - U_{i-1}D_i$, and we want to show that

- if $\mu \neq \lambda$, then the coefficient is 0;
- if $\mu = \lambda$, then the coefficient is 1.

Suppose $\mu \vdash i$, $\mu \neq \lambda$, and that μ can be obtained from λ by adding one box a to λ and then removing a box b (different from a) in $\lambda \cup \{a\}$, then this implies that b is a box already in λ and μ can also be obtained from λ by adding $(\lambda \setminus \{b\}) \cup \{a\}$. Therefore, the coefficient of any such μ is 0.

Suppose $\mu \vdash i$, $\mu \neq \lambda$, and that μ cannot be obtained from λ by adding one box and removing another different box, then μ can also not be obtained by removing one box and adding another different box from λ . Therefore, the coefficient of any such μ is 0.

Suppose $\mu = \lambda$. Then the coefficient of μ in $D_{i+1}U_i(\lambda)$ is $d(\lambda) + 1$, counting the number of ways to add a box to λ and then removing the same box from λ , and the coefficient of μ in $U_{i-1}D_i(\lambda)$ is $d(\lambda)$, counting the number of ways to remove a box from λ and adding the same box back to λ . Therefore, the coefficient of any such μ is 0. $\square$

Theorem 2.78 *The number of walks*

$$\alpha(D^n U^n, \emptyset) = n!$$

There are many proofs of this fact of different flavors:

- (1) Representation theoretic.
- (2) Linear algebraic.
- (3) Bijective.

Lemma 2.79 (Frobenius–Young identity)

$$D^n U^n \emptyset = n! \emptyset$$

Lemma 2.80 *For any polynomial or power series f with real coefficients in one variable,*

$$Df(UD)U = DUf(DU) = (UD + I)f(UD + I).$$

Proof. Since the operations D and U are linear, it suffices to check on the monomials $f(x) = x^k$:

$$Df(UD)U = D(UD)^k U = (DU)^{k+1} = DUf(DU) = (UD + I)f(UD + I),$$

where the last step is by Heisenberg commutation. $\square$

Lemma 2.81

$$D^n U^n = \prod_{i=1}^n (UD + iI).$$

Proof. Induction on n . When $n = 1$, $DU = UD + I$ is the Heisenberg.

$$\begin{aligned} D^{n+1} U^{n+1} &= D \cdot D^n U^n \cdot U \\ &= D \prod_{i=1}^n (UD + iI) U \\ &= \prod_{i=1}^{n+1} (UD + iI). \end{aligned}$$

$\square$

Proof of the Frobenius–Young identity. Since $D(\emptyset) = 0$,

$$D^n U^n(\emptyset) = \prod_{i=1}^n (UD + iI) \emptyset = n! \emptyset.$$

$\square$

Theorem 2.82 *Let $\lambda \vdash n$, and $\omega = A_m A_{m-1} \cdots A_1$ a valid λ word. Let*

$$D_w = \{i : A_i = D\}.$$

For each $i \in D_w$, let d_i be the number of D in w to the right of A_i , and let u_i be the number of U in w to the right of A_i . Therefore,

$$u_i - d_i = \text{the rank we arrive before taking the step } A_i = D.$$

Then

$$\alpha(w, \lambda) = f^\lambda \prod_{i \in D_w} (u_i - d_i).$$

Example 2.83 Suppose $w = U^3 D^2 U^2 D U^3$ and $\lambda = (2, 2, 1)$. Then $D_w = \{4, 7, 8\}$ and

$$(d_4, u_4) = (0, 3), (d_7, u_7) = (1, 5), (d_8, u_8) = (2, 5).$$

So the number of walks is

$$\alpha(w, \lambda) = 5(3 - 0)(5 - 1)(5 - 2) = 180.$$

Corollary 2.84

$$n! = \alpha(D^n U^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Proof. When $w = D^n U^n$, we have that $D_w = \{n + 1, n + 2, \dots, 2n\}$ and

$$(d_i, u_i) = (n, i - n - 1).$$

Therefore,

$$\alpha(w, \lambda) = f^\lambda \prod_{i \in D_w} (u_i - d_i) = f^\lambda n!$$

Therefore,

$$\alpha(D^n U^n, \emptyset) = f^\emptyset n! = n!$$

But on the other hand,

$$\alpha(D^n U^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

□

Lemma 2.85 *Under the same notation as the theorem, we prove that for any valid λ -word w ,*

$$w = \sum_{i-j=m} c_{ij}(w) U^i D^j$$

for some $c_{ij}(w) \in \mathbb{Z}$ and constant m uniquely determined by w .

Proof. There are two parts: first we show that w can be expressed as the desired sum, and second we show that the coefficients $c_{ij}(w)$ are uniquely determined by w .

For the first part, we use the lemma that $DU - UD = I$: if DU is in any word, then

$$DU = I + UD.$$

By induction, any w can be expressed as a sum of terms of $U^i D^j$, and $i - j$ is equal to the difference between the number of U and D 's in w , which is some constant m for every pair of i, j .

For the second part, uniqueness is equivalent to: if

$$\sum_{i-j=m} d_{ij} U^i D^j = 0$$

for some d_{ij} , then $d_{ij} = 0$ for every pair of i, j . Suppose for contradiction that not all d_{ij} 's are zero, and so there exists one that is nonzero. Let k be the minimal integer such that $d_{k+m,k} \neq 0$. Let $\mu \vdash k$, and let

$$\sum_{i-j=n} d_{ij} U^i D^j(\mu) = d_{k+m,k} U^{k+m} D^k(\mu) \neq 0,$$

a contradiction. □

Lemma 2.86 $DU^k = U^k D + kU^{k-1}$.

Proof. When $k = 0$. Both sides equal D , since $U^0 = I$.

$$DU^0 = D = U^0 D = D.$$

When $k = 1$, by the previous lemma:

$$DU = UD + I = U^1 D + 1 \cdot U^0.$$

Suppose as inductive hypothesis that $DU^k = U^k D + kU^{k-1}$ holds for some $k \geq 1$.

$$DU^{k+1} = D(U^k \cdot U) = (DU^k)U$$

(by inductive hypothesis)

$$= (U^k D + kU^{k-1})U$$

$$= U^k(DU) + kU^{k-1}U$$

(since $DU = UD + I$)

$$= U^k(UD + I) + kU^k$$

$$= U^{k+1}D + U^k + kU^k$$

$$= U^{k+1}D + (k+1)U^k.$$

□

Proof of Theorem 2.82. By the lemma, we have that

$$w = \sum_{i,j} c_{ij}(w) U^i D^j.$$

Then we follow the following steps:

(1) Apply U on the left of w :

$$Uw = \sum_{i,j} c_{ij}(w) U^{i+1} D^j.$$

By the uniqueness of the coefficients in any word in terms of $U^i D^j$,

$$c_{ij}(Uw) = c_{i-1,j}(w).$$

(2) Apply D on the left of w :

$$Dw = \sum_{i,j} c_{ij}(w) DU^i D^j.$$

By Theorem 2.86,

$$\sum_{i,j} c_{ij}(w) DU^i D^j = \sum_{i,j} c_{ij}(w) (U^i D + iU^{i-1}) D^j.$$

By the uniqueness of the coefficients in any word in terms of $U^i D^j$:

$$c_{i,j}(Dw) = r_{i,j-1}(w) + (i+1)c_{i+1,j}(w).$$

(3) Let w act on $\emptyset$ (since we are interested in $\alpha(w, \lambda)$).

$$w(\emptyset) = c_{n,0}U^n(\emptyset).$$

Therefore,

$$\alpha(w, \lambda) = c_{n,0}(w)\alpha(U^n, \lambda) = c_{n,0}f^\lambda.$$

(4) Compute $c_{n,0}$: Since

$$\begin{aligned} c_{i,j}(Dw) &= c_{i,j-1}(w) + (i+1)c_{i+1,j}(w), \\ c_{i,j}(Uw) &= c_{i-1,j}(w), \end{aligned}$$

setting $j = 0$ gives

$$\begin{aligned} c_{i,0}(Dw) &= (i+1)c_{i+1,0}(w), \\ c_{i,0}(Uw) &= c_{i-1,0}(w). \end{aligned}$$

This recursive formula allows us to compute that

$$c_{n,0}(w) = \prod_{k \in D_w} (u_k - d_k).$$

□

Remark 2.87 The last statement needs some justification: Since U lowers the index of i without multiplying, D raises the index of i with multiplying the current index, we end up taking the product of all the current levels at before each time we take D . Consider the following example $U^3 D^2 U^2 D U^3$, and $\lambda = (2, 2, 1)$. We compute $c_{5,0}$ by repeatedly applying the recursion:

$$\begin{aligned} c_{5,0}(U^3 D^2 U^2 D U^3) &= c_{4,0}(U^2 D^2 U^2 D U^3) \\ c_{4,0}(U^2 D^2 U^2 D U^3) &= c_{3,0}(U^1 D^2 U^2 D U^3) \\ c_{3,0}(U^1 D^2 U^2 D U^3) &= c_{2,0}(D^2 U^2 D U^3) \\ c_{2,0}(D^2 U^2 D U^3) &= 3 \cdot c_{3,0}(D U^2 D U^3) \\ 3 \cdot c_{3,0}(D U^2 D U^3) &= 3 \cdot 4 \cdot c_{4,0}(U^2 D U^3) \\ 3 \cdot 4 \cdot c_{4,0}(U^2 D U^3) &= 3 \cdot 4 \cdot c_{2,0}(D U^3) \\ 3 \cdot 4 \cdot c_{2,0}(D U^3) &= 3 \cdot 4 \cdot 3 c_{3,0}(U^3) \\ 3 \cdot 4 \cdot 3 c_{3,0}(U^3) &= 36 \cdot c_{0,0}(\emptyset) = 36. \end{aligned}$$

2.11. Robinson-Schensted-Knudsén Correspondence.

Observation 2.88 (a) When $n = 1$, we only have the SYT

$$\boxed{1}.$$

(b) When $n = 2$, we have the SYT

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \text{and} \quad \boxed{1} \boxed{2}.$$

(c) When $n = 3$, we have the SYT

$$\begin{array}{|c|c|c|}, & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \end{array}, & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}, & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\end{array}$$

(d) When $n = 4$, we have the SYT,

$$\begin{array}{|c|c|c|c|}, & \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \end{array}, & \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \end{array}, & \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \end{array}, & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, & \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, & \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}.\end{array}$$

Notice that in this case,

$$\begin{aligned} f^{\begin{array}{|c|} \hline 4 \\ \hline \end{array}} &= 1, f^{\begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array}} = 3, \\ f^{\begin{array}{|c|} \hline 2 \\ \hline 2 \\ \hline \end{array}} &= 2, f^{\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}} = 3, \\ f^{\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}} &= 1. \end{aligned}$$

We might observe that

$$4! = 24 = \sum_{\vdash n} (f^\lambda)^2 = 1^2 + 3^2 + 2^2 + 3^2 + 1.$$

Proposition 2.89 Given $n > 0$,

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

To prove this proposition, we establish a correspondence between symmetric group elements and standard Young tableaux.

Notation 2.90 Given a symmetric group S_n and an element σ , we can write σ in **one-line notation** as follows:

$$\sigma := \sigma(1) \sigma(2) \cdots \sigma(n).$$

Example 2.91 The symmetric group S_3 contains elements 123, 213, 132, 312, 231, 321.

Example 2.92 We can try using the one-line notation of S_3 to construct standard Young tableaux.

Consider $123 \in S_3$. We can put them in the Young diagram $\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$ in a row as they are to get the standard Young tableau $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$.

Consider the next element $213 \in S_3$. We can imitate the previous process to get a $\begin{array}{|c|} \hline 2 \\ \hline \end{array}$ when 2 comes in. Inserting 1, we have no choice but bump 2 to somewhere else. We can bump 2 to the right, or to the next row; we get to decide on how we do this, as long as we do it consistently and systematically. Let us bump 2 to the second row, and hence we obtain $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$. Inserting 3, we get $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \end{array}$.

Constructing standard Young tableaux using the above algorithm, for all S_3 elements, we have the following map from S_3 to the set of standard Young tableaux

of size 3:

$$\begin{aligned}
123 &\mapsto \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \\
213 &\mapsto \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \\
132 &\mapsto \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \\
321 &\mapsto \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \\
231 &\mapsto \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \\
312 &\mapsto \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}.
\end{aligned}$$

But the map is far from injective! The permutations 132 and 312 give the same standard Young tableaux under this algorithm. But something is different between them. If we track the steps in the construction of the SYT of 132, we have

$$132 \mapsto \boxed{1} \rightarrow \boxed{1 \ 3} \rightarrow \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}.$$

If we track the steps in the construction of the SYT of 312, we have

$$312 \mapsto \boxed{3} \rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}.$$

Observation 2.93 We have the following observations.

- (a) The permutations that result in the same SYT construct the SYT in distinctly different orders.
- (b) The arrangement of the boxes that record the location of i -th box being constructed is a SYT, for $i = 1, \dots, n$. For example, recording the location of the i -th box in the SYT of 132, we have $\begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}$; we have $\begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}$ for recording locations in the SYT of 312. If we pair these “location-recording SYT” with the previous set of SYT, we get a bijection between S_3 and the pairs of SYT of the same shape with size 3.

$$\begin{aligned}
123 &\mapsto \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\
213 &\mapsto \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \\
132 &\mapsto \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \\
321 &\mapsto \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
231 &\mapsto \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \\
312 &\mapsto \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}.
\end{aligned}$$

Proposition 2.94 (Robinson-Schensted Algorithm) *Given $n > 0$, we construct pairs of SYT for $s \in S_n$. Write $s = s_1 \cdots s_n$ using one-line permutation. Let us call the SYT on the left the **insertion tableau** and the SYT on the right the **recording tableaux**. Starting with $(\emptyset, \emptyset)$, insert number s_i in the one-line notation of s for each $i = 1, \dots, n$:*

- If s_i is bigger than all numbers in a row, append s_i at the right end of the row in the insertion tableau; append i at the right end on the same row in the recording tableau.
- If there exists a number that is bigger than s_i , replace the minimum of such number with s_i and insert the replaced number in the next row as above; append i at the same location as where s_i ends up.

Since s has finite length, this algorithm terminates and we obtain a pair of SYT of the same shape.

Theorem 2.95 (Robinson-Schensted Correspondence) *Given $n > 0$, the following insertion algorithm performed on the one-line notation of S_n gives a bijection between permutations in S_n and the pairs of standard Young tableaux of the same shape with size n .*

Corollary 2.96 *By RS Correspondence,*

$$n! = |S_n| = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Remark 2.97 The number f^λ for $\lambda \vdash n$ is the degrees of the irreducible representations of the symmetric group S_n , called the Specht module corresponding to λ . Our result is a special case that the sum of the squares of the degrees of the irreducible representations of a finite group G is equal to the order or size of the group G .

3. SIMPLICIAL COMPLEXES AND FACE ENUMERATION POLYNOMIALS

Definition 3.1 Let $V = \{x_1, \dots, x_n\}$ be a finite set, called a *vertex set*. An *abstract simplicial complex* on V is a collection Δ of subsets of V satisfying the following two conditions:

- (1) $\{x_i\} \in \Delta$, for $1 \leq i \leq n$.
- (2) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Notation 3.2 An element F of Δ is a *face*. A maximal face $\mathbb{F}$, a face that is not properly contained in any other face, is called a *facet*. The dimension of a face F is $|F| - 1$. We set $\dim(\emptyset) = -1$.

Definition 3.3 A *convex set* in $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ such that if $u, v \in S$, then the line segment joining u and v is also in S . The *convex hull* of a subset $S \subseteq \mathbb{R}^d$ is the intersection of all convex set in $\mathbb{R}^d$ containing S .

Definition 3.4 A set $\{v_0, v_1, \dots, v_j\} \subseteq \mathbb{R}^d$ is *affinely independent* if for any real numbers $\alpha_i \in \mathbb{R}^d$ with

$$\sum \alpha_i v_i = 0, \sum \alpha_i = 0,$$

then

$$\alpha_i = 0.$$

Definition 3.5 A *simplex* σ in $\mathbb{R}^d$ is the convex hull of an affinely independent subset $\mathbb{R}^d$. A *geometric simplicial complex* is a finite set Γ of simplices in $\mathbb{R}^d$ such that the following two conditions hold:

- (1) If $\sigma \in \Gamma$, and τ is a face of σ , then $\tau \in \Gamma$.
- (2) If $\sigma, \tau \in \Gamma$, then $\sigma \cap \tau$ is a common face of σ and τ .

To each simplicial complex Δ , we can associate with a geometric simplicial complex, called a geometric realization of Δ .

The following theorem is an analogue of the Whitney embedding theorem.

Theorem 3.6 (Karl Menger) *Any d -dimensional simplicial complex can be realized in $\mathbb{R}^{2d+1}$.*

This bound is sharp. For example, when $d = 1$, the complete graph K_5 and $K_{3,3}$ viewed as 1-dimensional simplicial complexes cannot be embedded in the 2-dimensional plane. This is a special case of the *forbidden minor theorem* on graph planarity.

Definition 3.7 For $i \geq -1$, let f_i be the number of i -dimensional faces of Δ . If $\dim \Delta = d - 1$, then

$$f(\Delta) = (f_0, f_1, \dots, f_{d-1})$$

is the f -vector of Δ .

Question 3.8 Given a sequence of d positive integers, when can this sequence be the f -vector of a simplicial complex?

Proposition 3.9 *Given positive integers n and j , there exist unique integers*

$$n_j > n_{j-1} > \dots > n_1 \geq 0$$

such that

$$n = \binom{n_j}{j} + \binom{n_{j-1}}{j-1} + \dots + \binom{n_1}{1}.$$

Proof. The proof consists of 2 parts. First, we establish the following identity: For $1 \leq i \leq m$,

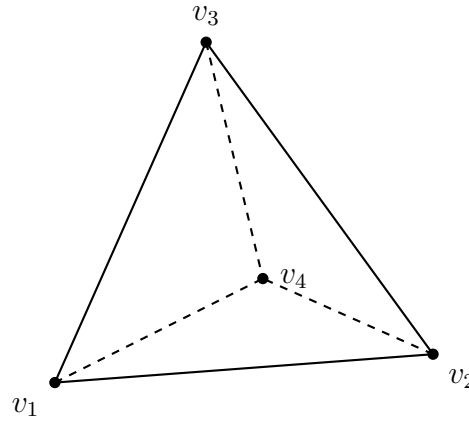
$$\binom{m}{i} + \binom{m-1}{i-1} + \dots + \binom{m-i+1}{1} + \binom{m-i}{0} = \binom{m+1}{i}.$$

The right hand side is the number of i -subsets in $[m+1]$, and the left hand side is the number of such subsets with the smallest number $k+1$ as k varies from 0 to i .

Second, we use this identity to prove the proposition by induction on j . Let $j = 1$, then

$$n = \binom{n_1}{1} = \binom{n}{1},$$

and so $n_1 = n$. Suppose the proposition is true for $j-1$. Given n, j , let m_j be the largest integer for which $n \geq \binom{m_j}{j}$. $\square$



Theorem 3.10 (Kruskal–Katona Theorem) *A vector $(f_0, f_1, \dots, f_{d-1}) \in \mathbb{P}^d$ is the f -vector of a $(d-1)$ -dimensional simplicial complex if and only if*

$$f_{i+1} \leq f_i^{(i+1)}, 0 \leq i \leq d-2.$$

Definition 3.11 Given the f -vector $(f_0, f_1, \dots, f_{d-1})$ of a $(d-1)$ -dimensional simplicial complex Δ , the h -vector of Δ is defined by the formula

$$f(\Delta; x-1) = \sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^i h_i x^{d-i}.$$

Example 3.12 (Boundary of a tetrahedron) The f -vector of the tetrahedron is $f(\Delta) = (1, 4, 6, 4)$.

$$\begin{aligned} h(\Delta; x) &= (x-1)^3 + 4(x-1)^2 + 6(x-1) + 4 \\ &= x^3 - 3x^2 + 3x - 1 \\ &\quad + 4x^2 - 8x + 4 \\ &\quad + 6x - 6 \\ &\quad + 4 \\ &= x^3 + x^2 + x + 1. \end{aligned}$$

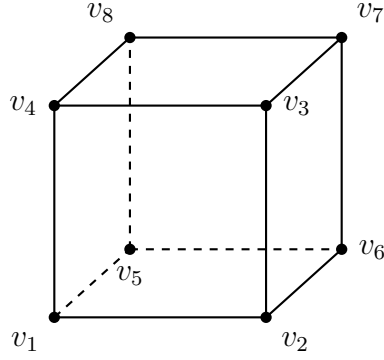
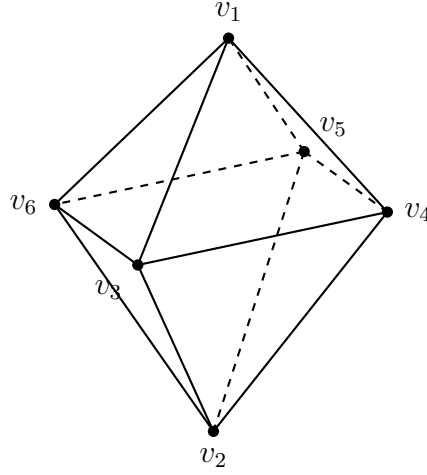
Therefore, the $h(\Delta)$ is $(1, 1, 1, 1)$.

Example 3.13 (Boundary of octahedron) The f -vector of the octahedron is

$$f(\Delta) = (1, 6, 8, 12).$$

Therefore, the f -polynomial is

$$f(\Delta; t) = \sum_{i=0}^d f_{i-1} t^{d-i}$$



and specializing to $(x - 1)$ gives

$$\begin{aligned}
 h(\Delta; x) &= (x - 1)^3 + 6(x - 1)^2 + 12(x - 1) + 8 \\
 &= x^3 - 3x^2 + 3x - 1 \\
 &\quad + 6x^2 - 12x + 6 \\
 &\quad + 12x - 12 \\
 &\quad + 8 \\
 &= x^3 + 3x^2 + 3x + 1.
 \end{aligned}$$

Therefore, the $h(\Delta) = (1, 3, 3, 1)$.

Example 3.14 (Boundary of an icosahedron)

$$\begin{aligned}
 h(\Delta; x) &= (x - 1)^3 + 12(x - 1)^2 + 30(x - 1) + 20 \\
 &= x^3 - 3x^2 + 3x - 1 \\
 &\quad + 12x^2 - 24x + 12 \\
 &\quad + 30x - 30 \\
 &\quad + 20 \\
 &= x^3 + 9x^2 + 9x + 1.
 \end{aligned}$$

The h -vector is $(h_0, h_1, h_2, h_3) = (1, 9, 9, 1)$.

Example 3.15 (Boundary of the cube) The f -vector of the boundary of the 3-cube is $(1, 8, 12, 6)$. Then

$$\begin{aligned} h(\Delta; x) &= (x-1)^3 + 8(x-1)^2 + 12(x-1) + 6 \\ &= x^3 - 3x^2 + 3x - 1 \\ &\quad + 8x^2 - 16x + 8 \\ &\quad + 12x - 12 \\ &\quad + 6 \\ &= x^3 + 5x^2 - x + 1. \end{aligned}$$

Proposition 3.16 *The relationship between h -vector and f -vectors can be seen through specializations.*

- (1) *Taking the coefficients of x^d on both sides, we see that $h_0 = 1$ unless $\Delta = \emptyset$.*
- (2) *Taking coefficients on both sides of x^{d-1} , we see that $h_1 = f_0 = d$.*
- (3) *If $x = 1$, then we obtain*

$$f_{d-1} = h_0 + h_1 + \cdots + h_d = \text{the number of } (d-1)\text{-dimensional faces}$$

- (4) *Rewriting the relationship between the f -vector and the h -vector gives*

$$(x-1)^{d-1} \sum_{i=0}^d f_{i-1} (x-1)^{1-i} = \sum_{i=0}^d h_i x^{d-i}.$$

Specializing to $x = 0$ gives

$$\begin{aligned} h_d &= (-1)^{d-1} \sum_{i=0}^d f_{i-1} (-1)^{i-1} \\ &= (-1)^{d-1} \sum_{i=0}^d f_{i-1} (-1)^{1-i} \\ &= (-1)^{d-1} (-f_{-1} + f_0 - \cdots + (-1)^{d-1} f_{d-1}). \end{aligned}$$

The quantity is called reduced Euler characteristic of Δ

$$\tilde{\chi}(\Delta) := -f_{-1} + f_0 - \cdots + (-1)^{d-1} f_{d-1},$$

which comes from the ordinary Euler characteristic of Δ

$$\chi(\Delta) := f_0 - \cdots + (-1)^{d-1} f_{d-1}.$$

Therefore,

$$h_d = (-1)^{d-1} \tilde{\chi}(\Delta).$$

This implies that the coefficient h_d is a topological invariant.

Remark 3.17 A triangulation of a topological space X is a simplicial complex Δ such that there is a geometric realization homeomorphic to X ,

$$|\Delta| \cong X.$$

The reason that our previous examples all have $h_d = 1$, is because they are all triangulations of the 2-dimensional sphere S^2 , whose topological Euler characteristic is 2.

Remark 3.18 If Δ is a simplicial sphere, i.e., a triangulation of the sphere S^d , then the Dehn–Sommerville equations imply that, for all $0 \leq i \leq d-1$,

$$h_i = h_{d-i}.$$

Polytopes are geometric objects with flat faces. A polytope is simplicial if all its faces are simplices. In general, the Dehn–Sommerville equations hold for all simplicial polytopes.

Theorem 3.19 *Let $F_1, \dots, F_t$ be a shelling of the simplicial complex Δ , with restrictions $G_1, \dots, G_t$. Then the generating polynomial of h admits a combinatorial interpretation*

$$\sum_{i=0}^d h_i x^i = \sum_{j=1}^t x^{|G_j|}.$$

If Δ is shellable, then h_i is the number of restrictions with i elements (independent of choice of shellings.)

Proof. We count the contribution of adjoining the j -th face F_j to the polynomial

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i}.$$

For the facet F_j , which has cardinality $|F_j| = d$, the unique minimal face G_j not contained in Δ_{j-1} satisfies $|G_j| = m$. Therefore, for each $i \geq 0$, there exist

$$\binom{d-m}{i}$$

faces in F_j with cardinality $m+i$ newly added to $\Delta_j \setminus \Delta_{j-1}$. Therefore, the contribution of these faces to the polynomial is

$$\begin{aligned} \sum_{i=0}^{d-m} \binom{d-m}{i} (x-1)^{d-(m+i)} &= \sum_{i=0}^{d-m} \binom{d-m}{d-m-i} (x-1)^{d-m-i} \\ &= \sum_{d-m-i=0}^{d-m} \binom{d-m}{d-m-i} (x-1)^{d-m-i} \\ &= ((x-1) + 1)^{d-m} \\ &= x^{d-m}. \end{aligned}$$

Therefore, summing over all contributions after adjoining all the facets, we have that

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{j=1}^t x^{d-|G_j|}$$

By the definition of the h -vector,

$$\sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{j=1}^t x^{d-|G_j|} = \sum_{i=0}^d h_i x^{d-i},$$

and therefore,

$$\sum_{j=1}^t x^{|G_j|} = \sum_{i=0}^d h_i x^i$$

□

Corollary 3.20 *If a (pure) $(d-1)$ -dimensional simplicial complex Δ is shellable, then $h_i(\Delta) \geq 0$ for every $0 \leq i \leq d$. If Δ is a triangulation of a topological space X and Δ is shellable, then*

$$h_d = (-1)^{d-1} \tilde{\chi}(X) \geq 0.$$

Example 3.21 Last time we have seen the two line segments, and considered two triangles. We will convince ourselves that neither of these are shellable by computing its h -vectors. Since the two triangles wedged at a vertex is a cone

$$C(\Delta) := \Delta \cup \{\{v\} \cup F \mid F \in \Delta\}.$$

You will prove in the homework that Δ is shellable if and only if the cone $C(\Delta)$ is shellable.

Example 3.22 If $F \in \Delta$, the link of F is the subcomplex

$$\text{lk}_\Delta(F) := \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

For example, the link of a vertex in the tetrahedron is the boundary of a triangle. If the homework you will be proving that Δ is shellable, then any link is shellable.

4. ENUMERATION OF GRAPHS UNDER GROUP ACTION

Let X be a finite set, and G be a subgroup of S_X . Let C be another set.

Definition 4.1 A *coloring* of X is a function $f: X \rightarrow C$.

We are interested in enumerating objects up to symmetry.

Definition 4.2 Two colorings f and g are G -equivalent, if there is an element $\pi \in G$ such that

$$g(\pi(x)) = f(x)$$

for all $x \in X$.

Before we do any counting, we need to know how many distinct objects/orbits/colorings there are in total. The following lemma gives us a count on the number of orbits: The average number of elements of Y fixed by an element of G is equal to the number of orbits.

Lemma 4.3 (Cauchy–Frobenius Lemma; Burnside’s Lemma) *Let Y be a finite set, and G be a subgroup of S_Y . For each $\pi \in G$, let*

$$\text{Fix}(\pi) = \{y \in Y \mid \pi(y) = y\}.$$

So $|Fix(\pi)|$ is the number of cycles of lengths one in the permutation π . Let Y/G be the set of orbits of G . Then

$$|Y/G| = \frac{1}{|G|} \sum_{\pi \in G} |Fix(\pi)|.$$

Example 4.4 Let Y be the colorings of the vertices of a square using black and white color. Let G be the group generated by a single rotation clockwise by 90 degrees

$$G = \{e, r, r^2, r^3\}.$$

For each element π in the group G , we count the number of colorings that are fixed by π , we look at which colorings are fixed.

- (1) For e , all the colorings are fixed by e .
- (2) For r and r^3 , only the following two colorings are fixed.

$$Fix(r) = Fix(r^3)$$

all white



all black



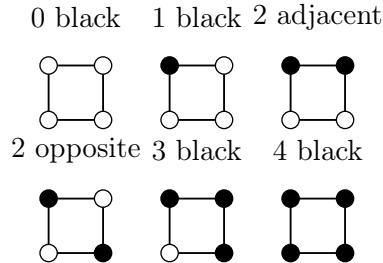
- (3) For r^2 , the following colorings are fixed.



Therefore, in total number of orbits is

$$|Y/G| = \frac{1}{4} (16 + 2 + 4 + 2) = 6,$$

which indeed correspond to the following distinct G -equivariant representatives.



Proof. For $y \in Y$, let the set $\text{Stab}_G(y)$ be the set of permutatitons fixing y :

$$\text{Stab}_G(y) = \{\pi \in G \mid \pi \cdot y = y\}.$$

Then the number on the right hand side of the equation is

$$\begin{aligned} \frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi)| &= \frac{1}{|G|} \sum_{\pi \in G} \sum_{y \in Y, \pi \cdot y = y} 1 \\ &= \frac{1}{|G|} \sum_{y \in Y} \sum_{\pi \in G, \pi \cdot y = y} 1 \\ &= \frac{1}{|G|} \sum_{y \in Y} |\text{Stab}_G(y)|. \end{aligned}$$

By Lemma 4.5, the above becomes

$$\begin{aligned} \frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi)| &= \frac{1}{|G|} \sum_{y \in Y} \frac{|G|}{|Gy|} \\ &= \sum_{y \in Y} \frac{1}{|Gy|}. \end{aligned}$$

Grouping the orbits together, we have that the above is

$$\begin{aligned} \frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi)| &= \sum_{y \in Y} \frac{1}{|Gy|} \\ &= \sum_O 1 \\ &= |Y/G|, \end{aligned}$$

where O ranges over all orbits of G on Y . □

Lemma 4.5 (Lemma for Burnside's Lemma) *The multiset of elements*

$$S = \{\pi \cdot y \mid y \in G\}$$

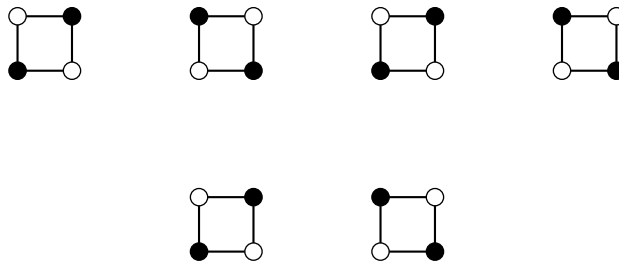
contains every element in the orbit

$$Gy = \{\pi \cdot y \mid y \in G\}$$

the same number of times, namely

$$\frac{|G|}{|Gy|} = |\text{Stab}_G(y)|.$$

Example 4.6 Let y be the coloring of vertices of a square with black and white colors such that its diagonals have different colors. The multiset S is equal to the following set: The orbit of y under G is equal to the following set:



Proof. Define $\varphi: G \rightarrow Gy$ by $\varphi(g) = g \cdot y$. By definition, φ is surjective. Our goal is to show that for any $z_1, z_2 \in Gy$, the preimage, called the *fiber* of z_1 and z_2 have the same size

$$|\varphi^{-1}(z_1)| = |\varphi^{-1}(z_2)|.$$

Since $z_1, z_2 \in Gy$, there exist $g_1, g_2 \in G$ such that

$$g_1 \cdot y = z_1, g_2 \cdot y = z_2.$$

We claim that the map $g_2 g_1^{-1}: \varphi^{-1}(z_1) \rightarrow \varphi^{-1}(z_2)$ is a bijection by proving that it is well-defined and has an inverse. For any $g \in \varphi^{-1}(z_1)$,

$$((g_2 g_1^{-1})g) \cdot y = g_2 g_1^{-1}(g \cdot y) = g_2 g_1^{-1} z_1 = g_2 g_1^{-1} g_1 y = g_2 \cdot y = z_2,$$

and $g_1 g_2^{-1}$ is its inverse.

Therefore, for every pair of elements z_1, z_2 , the fibers have the same size m . Since the cardinality of the multiset S is $|G|$ and its divided into $|Gy|$ copies of fibers of the same size m , the number m is

$$m = \frac{|G|}{|Gy|}.$$

In particular, to compute the size of every fiber, it suffices to compute the size of 1 special fiber – the fiber of y . By definition, $\varphi^{-1}(y)$ consists of all the elements in G such that $\varphi(g) = g \cdot y = y$:

$$\varphi^{-1}(y) = \text{Stab}_G(y) = \{\pi \in G \mid \pi \cdot y = y\},$$

and therefore,

$$\frac{|G|}{|Gy|} = |\text{Stab}_G(y)|.$$

□

Definition 4.7 Given a permutation $\pi \in S_X$, then π can be decomposed into distinct cycles, and let $c(\pi)$ be the number of cycles of π .

Example 4.8 Let σ be the permutation on 6 elements given by the one line notation

$$\sigma = 142536.$$

Then $\sigma = (1)(2453)(6)$, and $c(\sigma) = 3$.

Example 4.9 In the example where the group of rotations act on colorings of a square, we have that

$$e = (1)(2)(3)(4)$$

$$r = (1234)$$

$$r^2 = (13)(24)$$

$$r^3 = (1432)$$

Note that

$$6 = \frac{1}{4}(2^4 + 2^1 + 2^2 + 2^1) = \frac{24}{4},$$

where 2 is the number of colors and the exponents are the cycle numbers of each permutation.

Theorem 4.10 *Let G be the group of permutations of a finite set X . Then the number $N_G(n)$ of G -inequivalent n -colorings of X is given by*

$$N_G(n) = \frac{1}{|G|} \sum_{\pi \in G} n^{c(\pi)},$$

where $c(\pi)$ is the number of cycles in π .

Proof. For each $\pi \in G$, let π_n denote the action of $\pi \in G$ on the set C_n of n -colorings of X and C be the n colors. We want to determine the set of fixed points of π_n , so that we can apply Burnside's lemma. Let C be the set of n colors. If $f: X \rightarrow C$ is a coloring fixed by π , then for every $x \in X$,

$$f(x) = \pi_n \cdot f(x) = f(\pi(x)).$$

Therefore,

$$f \in \text{Fix}(\pi_n) \text{ if and only if } f(x) = f(\pi(x)) \text{ if and only if } f(x) = f(\pi^k(x)), k \geq 1.$$

Since the elements of X of the form $\pi^k(x)$ for $k \geq 1$ are exactly the elements in the cycle of π containing x , every $f \in \text{Fix}(\pi_n)$ has the property that all the elements within a cycle have the same color under f . In other words,

$$|\text{Fix}(\pi_n)| = n^{c(\pi)}.$$

Applying Burnside's lemma then gives the result. $\square$

We would like to count the total number of inequivalent colorings, but recording the number of occurrences of each color. This amounts to not refining the number $c(\pi)$ using cycle length.

Definition 4.11 Given a permutation π of an n -element set X , define the *type* of π as

$$\text{type}(\pi) = (c_1, \dots, c_n),$$

where π has c_i cycles of length i , with the property that

$$\sum_{i=1}^n i c_i = n.$$

Example 4.12 Let σ be the permutation on 6 elements given by the one line notation

$$\sigma = 142536.$$

Then $\sigma = (1)(2453)(6)$, and $c(\sigma) = 3$. The cycle type of σ is $(2, 0, 0, 1, 0, 0)$.

Definition 4.13 The cycle indicator of π is the monomial

$$Z_\pi = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}.$$

Given a subgroup G of S_X , the *cycle index polynomial* of G is

$$Z_G = Z_G(z_1, \dots, z_n) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi.$$

Example 4.14 Let σ be the permutation on 6 elements given by the one line notation

$$\sigma = 142536.$$

Then $\sigma = (1)(2453)(6)$, and $c(\sigma) = 3$. The cycle type of σ is $(2, 0, 0, 1, 0, 0)$. The cycle indicator is

$$Z_{142536} = z_1^2 z_4^1.$$

Example 4.15 The cycle indicator of the group of rotations acting on the vertices of a square is

$$Z_G = \frac{1}{4}(z_1^4 + z_2^2 + 2z_4).$$

Theorem 4.16 (Pólya's Theorem, 1937) *Let G be a group of permutations of the n -element set X . Let $C = \{r_1, r_2, \dots\}$ be a set of colors. Let $\kappa(i_1, i_2, \dots)$ be the number of inequivalent (under the action of G) colorings $f: X \rightarrow C$ such that color r_j is used i_j times. Let the generating polynomial of the number of G -inequivalent colorings such that the color r_j is used i_j times.*

$$F_G(r_1, r_2, \dots) = \sum_{i_1, i_2, \dots} \kappa(i_1, i_2, \dots) r_1^{i_1} r_2^{i_2} \cdots \in \mathbb{Z}[[r_1, r_2, \dots]]$$

Then we have the following formula for this generating function

$$\begin{aligned} F_G(r_1, r_2, \dots) &= Z_G(r_1 + r_2 + r_3 + \cdots, r_1^2 + r_2^2 + r_3^2 + \cdots, \dots, r_1^j + r_2^j + r_3^j + \cdots) \\ &= Z_G\left(\sum_{i=1}^{\infty} r_i^1, \sum_{i=1}^{\infty} r_i^2, \sum_{i=1}^{\infty} r_i^3, \dots, \sum_{i=1}^{\infty} r_i^j, \dots\right) \end{aligned}$$

Example 4.17 In the example where we have the the group of rotations acting on the vertices of a square, the cycle indicator of the group rotations is

$$Z_G = \frac{1}{4}(z_1^4 + z_2^2 + 2z_4).$$

Since we have 2 colors: black and white, the generating function is

$$F_G(b, w) = b^4 + b^3 w^1 + 2 \cdot b^2 w^2 + b^1 w^3 + w^4.$$

While the evaluation of the cycle indicator is

$$\begin{aligned} Z_G &= \frac{1}{4}((b+w)^4 + (b^2 + w^2)^2 + 2(b^4 + w^4)) \\ &= \frac{1}{4}(4b^4 + 4b^3 w^1 + 8b^2 w^2 + 4b^1 w^3 + 4w^4) \\ &= b^4 + b^3 w^1 + 2b^2 w^2 + b^1 w^3 + w^4. \end{aligned}$$

Indeed, $F_G(b, w) = Z_G(\sum_i r_i^j)$.

Example 4.18 In the example where we now change the group action on the vertices of the square to be all possible rotations and flips, the group elements

have the following cycles:

$$\begin{aligned} e &= (1)(2)(3)(4) \\ r &= (1234) \\ r^2 &= (13)(24) \\ r^3 &= (1432) \\ f &= (1)(23)(4) \\ fr &= (14)(23) \\ fr^2 &= (13)(2)(4) \\ fr^3 &= (13)(24) \end{aligned}$$

the cycle indicator of the group is

$$Z_G = \frac{1}{8} (z_1^4 + 3z_2^2 + 2z_2z_1^2 + 2z_4).$$

Since we have 2 colors: black and white, the generating function is

$$F_G(b, w) = b^4 + b^3w + 2 \cdot b^2w^2 + b^1w^3 + w^4.$$

While the evaluation of the cycle indicator is

$$\begin{aligned} Z_G &= \frac{1}{8} ((b+w)^4 + 3(b^2+w^2)^2 + 2(b^2+w^2)(b+w)^2 + 2(b^4+w^4)) \\ &= \frac{1}{8} (8b^4 + 8b^3w + 16b^2w^2 + 8bw^3 + 8w^4) \\ &= b^4 + b^3w + 2b^2w^2 + b^1w^3 + w^4. \end{aligned}$$

Indeed, $F_G(b, w) = Z_G(\sum_i r_i^j)$; but this time for a different group action: the group of rotations together with flips.

Example 4.19 (From necklaces to number theory) Recall that in the homework we have seen necklaces: a *necklace* of length ℓ is a curcular arrangement of ℓ (colored) beads. Two necklaces are the same if they are cyclic rotations of each other. We want to count the number of unique colorings of necklaces of length ℓ using Polya's enumeration. Let X be the set of ℓ (uncolored) beads

$$X = \{1, 2, \dots, \ell\}.$$

Let G be the cycle group of rotations of X , that is, if we write r for the rotation clockwise by $2\pi/\ell$ degrees, then

$$G = \{e, r, r^2, \dots, r^{\ell-1}\}.$$

For example, when $\ell = 4$, we recover our previous example of group rotations on a square.

For another example, when $\ell = 5$, we have

$$\begin{aligned} e &= (1)(2)(3)(4)(5) \\ r &= (12345) \\ r^2 &= (13524) \\ r^3 &= (14253) \\ r^4 &= (15432) \end{aligned}$$

The cycle indicator is

$$Z_G = \frac{1}{5}(z_1^5 + 4z_5^1)$$

For another example, when $\ell = 6$, we have

$$\begin{aligned} e &= (1)(2)(3)(4)(5)(6) \\ r &= (123456) \\ r^2 &= (135)(246) \\ r^3 &= (14)(25)(36) \\ r^4 &= (153)(264) \\ r^5 &= (165432) \end{aligned}$$

The cycle indicator is

$$Z_G = \frac{1}{6}(z_1^6 + 2z_6 + 2z_3^2 + z_2^3).$$

From this, we can observe and record the following facts and number theoretic tools:

- (1) The rotation r^m has $\gcd(m, \ell)$ cycles of length ℓ/d .
- (2) There is a number theoretic fact: for a fixed positive integer ℓ , an integer $1 \leq m \leq \ell$ has

$$\gcd(m, \ell) = d$$

if and only if

$$1 \leq m/d \leq \ell/d$$

and $\gcd(m/d, \ell/d) = 1$.

- (3) The number of such m/d is given by the Euler's totient function

$$\varphi(\ell/d) := |\{1 \leq i \leq \ell/d \mid \gcd(i, \ell/d) = 1\}| = \frac{\ell}{d} \prod_{p \mid (\ell/d)} \left(1 - \frac{1}{p}\right).$$

The cycle indicator is

$$Z_G(z_1, \dots, z_\ell) = \frac{1}{\ell} \sum_{m=1, \gcd(m, \ell)=d}^{\ell} z_{\ell/d}^d = \sum_{d \mid \ell} \varphi(\ell/d) z_{\ell/d}^d = \frac{1}{\ell} \sum_{d \mid \ell} \varphi(d) z_d^{\ell/d}.$$

Therefore, we have that the number of G -inequivalent necklaces is

$$N_\ell(n) = \frac{1}{\ell} \sum_{d \mid \ell} \varphi(\ell/d) n^d.$$

If $n = 1$, then

$$N_\ell(1) = 1 = \frac{1}{\ell} \sum_{d|\ell} \varphi(\ell/d) \iff \sum_{d|\ell} \varphi(d) = \ell.$$

Proof of the Pólya theorem. Let $t = |X|$ and $i_1 + i_2 + \dots = t$, where each $i_j \geq 0$. For $\mathbf{i} = (i_1, i_2, \dots) \in \mathbb{Z}_{\geq 0}^n$ such that $\sum i_j = t$, let $C_{\mathbf{i}}$ denote the set of all colorings of X with color r_j used i_j times. The group G acts on $C_{\mathbf{i}}$. The group G acts on the set of all colors:

- (1) If $e \in G$ is the identity, then $e \cdot f = f \in C_{\mathbf{i}}$.
- (2) If $f \in C_{\mathbf{i}}$ and $g \in G$, then $g \cdot f \in C_{\mathbf{i}}$.
- (3) If $g, g' \in G$ and $f \in C_{\mathbf{i}}$, then $(g \circ g') \cdot f = g \cdot (g' \cdot f) \in C_{\mathbf{i}}$.

We will use Burnside Lemma, or Cauchy–Frobenius Lemma to prove that the Let $g_{\mathbf{i}}$ denote the action of an element $g \in G$ on $C_{\mathbf{i}}$. We will count $|\text{Fix}(\pi_{\mathbf{i}})|$.

The set

$$\text{Fix}(\pi_{\mathbf{i}}) = \{f \in C_{\mathbf{i}} \mid f \text{ satisfies the following conditions (a) and (b)}\},$$

where the conditions are as follows:

- (a) In any cycle of π , all the elements in the same cycle gets the same color.
- (b) the color r_j appears i_j times.

Consider the following product

$$H_\pi = \prod_j \left(\sum_i r_i^j \right)^{c_j(\pi)}$$

where $c_j(\pi)$ is the number of cycles of length j of π . Expanding this product as a sum of monomials

$$\prod_i r_i^{j_i},$$

we get one of these monomials by doing the following two steps:

- choosing a term r_k^j from each factor of H_π and multiplying; choosing the term r_k^j corresponds to coloring all the elements of some length- j cycle with the color r_k . In addition, since every factor

$$r_1^j + r_2^j + \dots$$

occurs exactly $c_j(\pi)$ times in H_π , each factor corresponds a coloring where all elements in every cycle have the same color.

- multiplying these terms together gives the monomial

$$r_1^{j_1} r_2^{j_2} \dots$$

which corresponds to using the color r_k a total of j_k times.

Therefore, the coefficient of the monomial

$$\prod_i r_i^{j_i}$$

is equal to $|\text{Fix}(\pi_{\mathbf{i}})|$, and so

$$H_\pi = \sum_{\mathbf{i}} |\text{Fix}(\pi_{\mathbf{i}})| r_1^{i_1} r_2^{i_2} \dots$$

Taking the average of H_π over G using the left hand side:

$$\frac{1}{|G|} \sum_{\pi \in G} H_\pi = \frac{1}{|G|} \sum_{\pi \in G} \prod_j \left(\sum_i r_i^j \right)^{c_j(\pi)} = Z_G \left(\sum_{i=1}^{\infty} r_i^1, \sum_{i=1}^{\infty} r_i^2, \sum_{i=1}^{\infty} r_i^3, \dots, \sum_{i=1}^{\infty} r_i^j, \dots \right).$$

Taking the average of H_π over G using the right hand side:

$$\begin{aligned} \frac{1}{|G|} \sum_{\pi \in G} \sum_{\mathbf{i}} |\text{Fix}(\pi_{\mathbf{i}})| r_1^{i_1} r_2^{i_2} \dots &= \sum_{\mathbf{i}} \frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi_{\mathbf{i}})| r_1^{i_1} r_2^{i_2} \dots \\ &= \sum_{\mathbf{i}} \left(\frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi_{\mathbf{i}})| \right) r_1^{i_1} r_2^{i_2} \dots \\ &= F_G(r_1, r_2, \dots) \end{aligned}$$

where we used the equality that

$$\frac{1}{|G|} \sum_{\pi \in G} |\text{Fix}(\pi_{\mathbf{i}})| = \kappa(i_1, i_2, \dots)$$

by the Burnside's Lemma (Cauchy–Frobenius Lemma). $\square$

Example 4.20 Let $G = S_\ell$, the group of all permutations of $\{1, 2, \dots, \ell\} = X$. For example,

$$\begin{aligned} Z_{S_3}(z_1, z_2, z_3) &= \frac{1}{6}(z_1^3 + 3z_1z_2 + 2z_3) \\ Z_{S_4}(z_1, z_2, z_3, z_4) &= \frac{1}{24}(z_1^4 + 6z_1^2z_2 + 3z_2^2 + 8z_1z_3 + 6z_4). \end{aligned}$$

For each frequency vector $\mathbf{i} = (i_1, i_2, \dots)$, we want to count the number of inequivalent colorings in $C_{\mathbf{i}}$. We have the following two facts:

- If two colorings f, f' of X use each color the same number of times, then there is some permutation of X that sends one of the colorings to the other. Therefore, $C_{\mathbf{i}}$ consists of a single orbit, and so

$$\kappa(\mathbf{i}) = \kappa(i_1, i_2, \dots) = 1.$$

- By construction,

$$i_1 + i_2 + \dots = \ell.$$

Therefore, the number of S_ℓ -inequivalent colorings on X such that the color r_j is used i_j times is

$$F_{S_\ell}(r_1, r_2, \dots) = \sum_{i_1 + i_2 + \dots = \ell} r_1^{i_1} r_2^{i_2} \dots$$

Note that we have the following identity of power series, by the lemma

$$\sum_{\ell \geq 0} F_{S_\ell}(r_1, r_2, \dots) x^\ell = \sum_{\ell \geq 0} \left(\sum_{i_1 + i_2 + \dots = \ell} r_1^{i_1} r_2^{i_2} \dots \right) x^\ell = \frac{1}{(1 - r_1 x)(1 - r_2 x) \dots}.$$

For each fixed n and n distinct colors, if $f_n(\ell)$ is the number of S_ℓ -inequivalent n -colorings of X , then

$$\sum_{\ell \geq 0} f_n(\ell) x^\ell = \frac{1}{(1 - x)^n}.$$

Furthermore, we have the identity

$$\frac{1}{(1-x)^n} = \sum_{\ell \geq 0} \binom{n+\ell-1}{\ell} x^\ell$$

using the *negative binomial expansion*

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

This implies the number of inequivalent n -colorings of a set of size ℓ is

$$f_n(\ell) = \binom{n+\ell-1}{\ell},$$

which is the number of distinct ways to pick ℓ things with repetition from n things.

Lemma 4.21 For each r and x ,

$$\frac{1}{(1-rx)} = \sum_{j \geq 0} (rx)^j.$$

Example 4.22 Recall that the number of G -inequivalent n -colorings of X for any group G of permutations of X is given by

$$N_G(n) = \frac{1}{|G|} \sum_{\pi \in G} n^{c(\pi)}.$$

When $G = S_\ell$, the above example shows us that

$$\frac{1}{\ell!} \sum_{\pi \in S_\ell} n^{c(\pi)} = \binom{n+\ell-1}{\ell} = \frac{1}{\ell!} n(n+1)(n+2) \cdots (n+\ell-1).$$

Equivalently, if $c(\ell, k)$ is the number of permutations in S_ℓ with k cycles (*unsigned Stirling number of the first kind*), then the generating polynomial of $c(\ell, k)$ is

$$\sum_{k=1}^{\ell} c(\ell, k) x^k = x(x+1)(x+2) \cdots (x+\ell-1).$$

This implies that this generating polynomial is a real-rooted polynomial, and its coefficients $c(\ell, k)$ form a strongly log-concave sequence.

In the example before, we were able to compute the generating functions F_G without computing the Z_G . But we could still compute the Z_G functions in this case.

Theorem 4.23 Let $\sum_i c_i = \ell$. Then the number of permutations $\pi \in S_\ell$ with c_i cycles of length i (i.e., the coefficient of $z_1^{c_1} z_2^{c_2} \cdots$ in the function $\ell! Z_{S_\ell}(z_1, \dots, z_\ell)$) is equal to

$$\frac{\ell!}{(1^{c_1} \cdot c_1!)(2^{c_2} \cdot c_2!)(3^{c_3} \cdot c_3!) \cdots}.$$

Proof. For each cycle type vector $\mathbf{c} = (c_1, c_2, \dots)$, we define

$X_{\mathbf{c}} :=$ set of all permutations $\pi \in S_\ell$ with c_i cycles of length i .

For any well-defined map

$$f: S_\ell \rightarrow X_{\mathbf{c}}, \pi_1 \pi_2 \cdots \pi_\ell \mapsto \prod_{j=1}^{c_1} (\pi_j) \prod_{j=1}^{c_2} (\pi_{c_1+2j-1} \pi_{c_1+2j}) \prod_{j=1}^{c_3} (\pi_{c_1+2c_2+3j-2} \pi_{c_1+2c_2+3j-1} \pi_{c_1+2c_2+3j}) \cdots$$

To count the number of $X_{\mathbf{c}}$, we need to count the preimage of each π under f . Each cycle of length i can be written in i ways, and there are $c_i!$ ways of ordering c_i cycles of length i . Therefore,

$$|f^{-1}(\pi)| = \prod_{i=1} (c_i!)(i^{c_i}).$$

In particular, all the fibers have the same size. Therefore

$$|X_{\mathbf{c}}| = \frac{|S_\ell|}{\prod_{i=1} (c_i!)(i^{c_i})},$$

as desired. $\square$

Using this theorem, we can prove the following.

Theorem 4.24 *The generating function of all Z_{S_ℓ} for $\ell \geq 0$ is*

$$\sum_{\ell \geq 0} Z_{S_\ell}(z_1, z_2, \dots) x^\ell = \exp(z_1 x + z_2 \frac{x^2}{2} + \cdots).$$

Proof. The right hand side

$$\begin{aligned} \exp(z_1 x + z_2 \frac{x^2}{2} + z_3 \frac{x^3}{3} \cdots) &= \exp(z_1 x) \exp(z_2 \frac{x^2}{2!}) \exp(z_3 \frac{x^3}{3}) \cdots \\ &= \left(\sum_{n \geq 0} \frac{(z_1 x)^n}{n!} \right) \left(\sum_{n \geq 0} \frac{(z_2 x^2)^n}{2^n n!} \right) \left(\sum_{n \geq 0} \frac{(z_3 x^3)^n}{3^n n!} \right) \cdots \end{aligned}$$

The coefficient of $z_1^{c_1} z_2^{c_2} \cdots x^\ell$ for $\ell = \sum_i i c_i$ is

$$\frac{1}{\prod_i (c_i!)(i^{c_i})} = \frac{1}{\ell!} \frac{\ell!}{\prod_i (c_i!)(i^{c_i})},$$

which is the coefficient of $z_1^{c_1} z_2^{c_2} \cdots$ in Z_{S_ℓ} . $\square$

As a sanity check of Pólya's Theorem, we set $z_j = n = \sum_i r_i^j$ where $r_i = 1$ for all i , we obtain

$$\begin{aligned} \sum_{\ell \geq 0} Z_{S_\ell}(n, n, \dots) x^\ell &= \exp \left(nx + n \frac{x^2}{2} + n \frac{x^3}{3} + \cdots \right) \\ &= \exp(-n \log(1 - x)) \\ &= \frac{1}{(1 - x)^n}, \end{aligned}$$

using the identity

$$\log(1 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

Example 4.25 (Revisiting quotient of Boolean algebras) Recall that if X is a finite set, B_X is the Boolean algebra of all subsets of X , and G is a group of permutation of X . We want to count the number of elements in the quotient B_X/G in level i , which is the same as the number of G -inequivalent colorings of X to the color set $\{0, 1\}$, since

$$s \in S_f \iff f(s) = 1.$$

For any two 2-colorings f and f' , they are G -equivalent if and only if S_f and $S_{f'}$ are in the same orbit of G . Therefore, the number of inequivalent 2-colorings f of X with i values equal to 1 is $|B_X/G|_i$. Then, $\kappa(i, j)$ is the number of G -inequivalent 2-colorings of X with the colors 0 and 1 such that 0 is used j times and 1 is used i times with $i + j = |X|$. Then Pólya's Theorem, we have that

$$\sum_{i,j} \kappa(i, j) x^i y^j = Z_G(x + y, x^2 + y^2, x^3 + y^3, \dots)$$

(i) If X is a 2×2 rectangle, and $G = S_2 \wr S_2$, and

$$G = \{(1)(2)(3)(4), (12)(3)(4), (1)(2)(34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}.$$

Then

$$Z_G(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4).$$

(ii) From the formula,

$$\sum_{i=0}^4 |P_i| q^i = \frac{1}{8}((1+q)^4 + 2(1+q)^2(1+q^2) + 3(1+q^2)^2 + 2(1+q^4)) = 1 + q + 2q^2 + q^3 + q^4.$$

This number is

$$1 + q + 2q^2 + q^3 + q^4 = \left[\begin{matrix} |X| \\ 2 \end{matrix} \right]_q = \left[\begin{matrix} 4 \\ 2 \end{matrix} \right]_q.$$

This is consistent with what we proved earlier: If $p_i(m, n)$ is the number of elements of $L(m, n)$ of rank i , then the generating function

$$\sum_{i \geq 0} p_i(m, n) q^i = \left[\begin{matrix} m + n \\ m \end{matrix} \right]_q.$$

(ii) If X is the graph K_4 , and G is the S_6 action on the edges, then we have The

Cycles of lengths	Permutation	Counts
(1, 1, 1, 1, 1, 1)	(12)(13)(14)(23)(24)(34)	1
(2, 2, 1, 1)	(12)(13, 23)(14, 24)(34)	6
(3, 3)	(12, 23, 13)(14, 24, 34)	8
(2, 2, 1, 1)	(12)(13, 24)(14, 23)(34)	3
(4, 2)	(12, 23, 34, 14)(13, 24)	6

cycle index polynomial is

$$Z_G(z_1, \dots, z_6) = \frac{1}{24}(z_1^6 + 9z_1^2 z_2^2 + 8z_3^2 + 6z_2 z_4).$$

Specializing $z_i = 1 + q^i$ gives us

$$\sum_{i=0}^6 |P_i| q^i = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6.$$

The theory of rank-symmetry, rank-unimodality and Sperner properties of these posets tell us that the polynomials Z_G 's are palindromic and unimodal. But for both examples, they fail to be log-concave.

REFERENCES

- [HJ12] Roger A Horn and Charles R Johnson, *Matrix analysis*, Cambridge university press, 2012. [9](#), [11](#), [13](#), [25](#)

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