

CHOW AND CHAIN POLYNOMIALS OF MATROIDS ARE NOT LOG-CONCAVE

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ABSTRACT. The Hilbert–Poincaré series of the Chow rings of the toric varieties of matroids, often called the *Chow polynomials*, and the f -polynomial of their boundary complexes, called the *chain polynomials*, have been conjectured to only have real zeros. We construct several explicit infinite families of matroids from finite geometries, whose multichain counts fail to be log-concave. They share a common geometric construction called *hyperplane coalescence*, together with several classical operations in projective geometry and combinatorial designs. This induces failures of log-concavity for Chow, augmented Chow, chain, and h -polynomials of several explicit infinite families of matroids, disproving the real-rootedness conjectures of Ferroni–Schröter, Huh–Stevens, and Athanasiadis–Kalampongia–Evangelinou from the lowest possible ranks.

As an application of the same mechanism, we also give counterexamples to the log-concavity of h -polynomials of graph associahedra, answering in the negative a question of Sack.

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1. INTRODUCTION

Let M be a loopless matroid of positive rank r . In [FY04], Feichtner and Yuzvinsky introduced the Chow ring $\underline{\text{CH}}(M)$ as the Chow ring of the toric variety associated to the Bergman fan of M . This toric variety has a boundary complex of torus-equivariant subvarieties isomorphic to the order complex $\underline{\Delta}(M)$ of the lattice of flats of M . The *Chow polynomial* and *chain polynomial* of M are, respectively, the Hilbert–Poincaré polynomial of $\underline{\text{CH}}(M)$ and the face polynomial of this boundary complex:

$$\underline{H}(M; t) = \sum_{i=0}^{r-1} \dim \underline{\text{CH}}^i(M) t^i, \quad f(M; t) := \sum_{i=0}^r f_{i-1}(\underline{\Delta}(M)) t^i.$$

Later in [BHM⁺22], Braden, Huh, Matherne, Proudfoot and Wang introduced the toric variety of the augmented Bergman fan of M , and defined the augmented Chow

ring $\text{CH}(\mathbf{M})$ as its Chow ring. Its Hilbert–Poincaré polynomial is the *augmented Chow polynomial*:

$$H(\mathbf{M}; t) = \sum_{i=0}^r \dim \text{CH}^i(\mathbf{M}) t^i.$$

These polynomial invariants of the toric varieties of matroids only depend on the underlying matroids, and have been proven to enjoy many positivity properties, coming from combinatorial Hodge theory, and the structural properties of the boundary complexes. In the seminal paper by Adiprasito, Huh and Katz [AHK18], the Chow ring of a matroid, despite being the Chow ring of a possibly incomplete toric variety, satisfies the Kähler package, similar to the cohomology ring of a smooth projective variety. Furthermore, Braden, Huh, Matherne, Proudfoot and Wang proved the corresponding statements for the augmented Chow ring [BHM⁺22, Theorem 1.6]. In particular, as a result of Poincaré duality and the hard Lefschetz theorem, the Hilbert–Poincaré series $\underline{H}(\mathbf{M}; t)$ and $H(\mathbf{M}; t)$ have symmetric, unimodal coefficients.

For the chain polynomials, the boundary complex $\underline{\Delta}(\mathbf{M})$ is well-known to be vertex decomposable, and hence shellable. This implies that the sequence of face numbers $f_i(\underline{\Delta}(\mathbf{M}))$ is strongly flawless, and that the h -vector of $\underline{\Delta}(\mathbf{M})$ is nonnegative [BW96]. Coron, Ferroni and Li studied structural properties of the nested set complexes in the most general setting of arbitrary matroids with building sets [CFL26b]. They proved that reduced nested set complexes of matroids with arbitrary building sets are vertex decomposable, hence shellable, and admit convex ear decompositions [CFL26b, Corollary 1.2 and Theorem 1.3]. The existence of convex ear decompositions, for the maximal building sets, implies that the h -polynomial of the order complex $\underline{\Delta}(\mathbf{M})$, written as $h(\mathbf{M}; t) = \sum_{i=0}^d h_i t^i$ with $d = r - 1$ is strongly flawless: for every $0 \leq i \leq \lfloor d/2 \rfloor$,

$$h_0 \leq h_1 \leq \dots \leq h_{\lfloor d/2 \rfloor}, \quad h_i \leq h_{d-i}.$$

In particular, it is top-heavy [CFL26b, Corollary 1.4].

However, stronger positivity properties of the Hilbert–Poincaré series $\underline{H}(\mathbf{M}; t)$ and $H(\mathbf{M}; t)$ need to come beyond the Kähler package; and similarly for the face enumeration polynomials $f(\mathbf{M}; t)$ and $h(\mathbf{M}; t)$, they must come beyond the structural properties mentioned above. For instance, Ferroni, Matherne, Stevens and Vecchi proved γ -positivity for both polynomials $\underline{H}(\mathbf{M}; t)$ and $H(\mathbf{M}, t)$ [FMSV24] as an immediate application of the semi-small decomposition of the Chow ring and the augmented Chow ring [BHM⁺22].

Real-rootedness of a polynomial with nonnegative real coefficients is widely regarded as the strongest positivity property. It is equivalent to the total nonnegativity of the infinite Töeplitz matrix associated to the polynomial, and implies log-concavity without internal zeros by a classical result of Newton [New07]. For palindromic polynomials, it implies γ -positivity. We refer to the beautiful handbook by Brändén on the relationships among these positivity properties [Brä15].

Along this line, Huh and Stevens, and independently Ferroni and Schröter, proposed the following conjecture for the Chow polynomials of matroids.

Conjecture 1.1 ([Ste21, Conjecture 4.1.3] [FS24, Conjecture 8.18]) For any loopless matroid $\mathbf{M}$, the polynomial $\underline{H}(\mathbf{M}; t)$ is real-rooted.

Huh and Stevens proposed the augmented analogue [Ste21, Conjecture 4.3.3]; see also [FMSV24, Conjecture 1.6].

Conjecture 1.2 (Huh–Stevens) For any loopless matroid M , the polynomial $H(M; t)$ is real-rooted.

Around the same time, Athanasiadis and Kalampoglia-Evangelinou conjectured the following analogous statement [AKE23, Conjecture 1.2].

Conjecture 1.3 (Athanasiadis–Kalampoglia-Evangelinou) For any loopless matroid M , the polynomial $f(M; t)$ is real-rooted.

These conjectures have stimulated much research activity in recent years, and they are proven to hold for several prominent classes of matroids. For chain polynomials, Athanasiadis and Kalampoglia-Evangelinou proved real-rootedness of chain polynomials for subspace lattices, partition lattices and their type B analogues, as well as the uniform matroids and near-pencil cases [AKE23, Theorem 1.3]. Brändén and Leite established it for perfect matroid designs, Dowling geometries and paving matroids [BL25], and for all matroids of rank at most 3 [BL25, Theorem 6.1].

For Chow polynomials and the augmented Chow polynomials, their real-rootedness for uniform matroids were first proved [FMSV24, Theorem 1.10] and by Brändén and Vecchi [BV25b, Theorem 1.1]. Soon after, Hoster and Stump generalized both results to finite graded simplicial posets with nonnegative h -vectors [HS25, Theorem 1.1]. Brändén and Vecchi also proved real-rootedness of Chow and augmented Chow polynomials for totally nonnegative posets and their duals, and for paving matroids [BV25a, Theorem 6.1 and Corollary 7.2].

Concurrently, Coron, Ferroni and Li introduced UMEL-shellable posets, which are posets admitting an EL-labeling satisfying uniformity and monotonicity conditions. They proved real-rootedness of these polynomials for UMEL-shellable posets [CFL25, Theorems 1.2 and 1.5]. For matroids, UMEL-shellable posets include the lattice of flats of prominent families of matroids such as uniform matroids, affine and projective geometries, braid matroids of types A and B, all Dowling geometries, and more generally rank uniform supersolvable matroids, together with their truncations and rank-selections [CFL25, Theorem 1.6, Theorem 1.8 and Proposition 7.1]. In recent work on matroid analogues of Gal’s conjecture, Coron, Ferroni and Li further proved that $\underline{H}(M; t)$ is real-rooted for every matroid of rank at most 6 [CFL26a, Corollary 8.5].

The main goal of this paper is to construct explicit infinite families of counterexamples to all the conjectures above. They share a common source which comes from the failure of log-concavity of flag h -vectors. This failure can also be rephrased into the failure of log-concavity of multichain counts of matroids. The *multichain counts* of a loopless matroid M are defined by the function $\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$

$$Z_M(k) := \#\{\widehat{0} = F_0 \leq F_1 \leq \dots \leq F_k = \widehat{1} \in \mathcal{L}(M)\},$$

which counts the multichains of $\mathcal{L}(M)$ by size. See [Sta12, Section 3.12]. It turns out that, the failure of log-concavity of the multichain counts, induces failures of log-concavity of the polynomial invariants in question; this idea appeared implicitly in the recent work of Luis Ferroni [Fer26] and in an analogous setting in Ehrhart theory [FH24, Conjecture 1.2].

Theorem A *Let M be a loopless matroid. If there is an integer $j \geq 2$ such that*

$$Z_M(j)^2 < Z_M(j-1)Z_M(j+1),$$

then for every sufficiently large integer n , the polynomials

$$f(M^{\oplus n}; t), \quad h(M^{\oplus n}; t), \quad \underline{H}(M^{\oplus n}; t), \quad H(M^{\oplus n}; t)$$

are not log-concave, and hence not real-rooted.

Our original mechanism for failure of log-concavity of the polynomials in question uses the failure of log-concavity of flag h -vectors and flag f -vectors similar to the statement above, instead of failures of multichain counts. The statement of the above theorem is then suggested by Ferroni, after we communicated the main discovery below, which simplifies our proofs.

We construct four families of matroids from finite geometries whose multichain counts satisfy the hypothesis of Theorem A. All four families use a geometric technique which we call *hyperplane coalescence*. In a nutshell, hyperplane coalescence merges selected collections of hyperplanes into larger hyperplanes while preserving every flat of lower rank, enabling us to alter chain counts without changing the lower-rank geometry of the matroid. Hyperplane coalescence generalizes the classical operation of tightening a free basis, which is the inverse of *circuit-hyperplane relaxation*, and can be viewed as a matroid operation that naturally arises in the Crapo–Knuth theory of detruncations [Cra70, Knu75]; see also [PvdP17, Section 3].¹

- (1) The *affine Klein family* M_q of rank 4, for odd prime powers q , constructed from affine geometry using spreads obtained from the Klein correspondence [Hir85].
- (2) The *affine packing family* M_q^{pk} of rank 4, for prime powers q , constructed from affine geometry using Denniston’s packings of projective lines into spreads [Den72, Den73].
- (3) The *Gabidulin family* $G_{k,q}$ of rank k , for $k \geq 4$ and prime powers q , constructed from projective geometry using rank-metric codes [Del78, Gab85].
- (4) The *projective family* $C(q, N)$ of rank 7, for prime powers q and integers $4 \leq N \leq q^3$, constructed from projective planes and auxiliary four-element sets using rank-distance bounds [Del78, Gab85].

Theorem B *The following equivalences hold:*

- (a) *For every odd prime power q ,*

$$Z_{M_q}(3)^2 < Z_{M_q}(2)Z_{M_q}(4) \iff q \geq 97.$$

- (b) *For every prime power q ,*

$$Z_{M_q^{\text{pk}}}(3)^2 < Z_{M_q^{\text{pk}}}(2)Z_{M_q^{\text{pk}}}(4) \iff q \geq 81.$$

- (c) *For every prime power q ,*

$$Z_{G_{4,q}}(3)^2 < Z_{G_{4,q}}(2)Z_{G_{4,q}}(4) \iff q \geq 81.$$

- (d) *For every prime power q ,*

$$Z_{C(q,q^3)}(3)^2 < Z_{C(q,q^3)}(2)Z_{C(q,q^3)}(4) \iff q \geq 149.$$

¹Throughout this paper, we use *detruncation* for what is classically called a nontrivial *erection*: Given a matroid M , a *detruncation* of M is a rank- r matroid N whose rank- $(r-1)$ truncation is M . Our terminology emphasizes its relation to truncation.

For each matroid M satisfying one of these inequalities and every sufficiently large integer n , the polynomials

$$f(M^{\oplus n}; t), \quad h(M^{\oplus n}; t), \quad \underline{H}(M^{\oplus n}; t), \quad H(M^{\oplus n}; t)$$

are neither log-concave nor real-rooted.

Example 1.4 The matroid M_{97} is on 88 533 986 elements of rank 4, with

$$\begin{aligned} Z_{M_{97}}(2) &= 981\,796\,552\,261\,343, \\ Z_{M_{97}}(3) &= 28\,747\,469\,795\,558\,806, \\ Z_{M_{97}}(4) &= 898\,967\,215\,589\,248\,050. \end{aligned}$$

These numbers of multichains satisfy

$$\frac{Z_{M_{97}}(3)^2}{Z_{M_{97}}(2)Z_{M_{97}}(4)} \approx 0.93634 < 1.$$

The chain polynomial of the full lattice $\mathcal{L}(M_{97})$ is

$$\begin{aligned} f(\mathcal{L}(M_{97}); t) &= 1 + 981\,796\,552\,261\,343t + 27\,765\,673\,243\,297\,463t^2 \\ &\quad + 842\,454\,072\,550\,391\,781t^3 + 1\,605\,538\,311\,579\,936\,540t^4 \\ &\quad + 789\,868\,115\,720\,580\,880t^5, \end{aligned}$$

with the coefficients a_1, a_2, a_3 of the terms t, t^2, t^3 satisfying

$$a_2^2 - a_1a_3 = -56\,185\,893\,214\,937\,313\,548\,622\,648\,985\,514 < 0.$$

The matroid $N = M_{97} \oplus M_{97}$ has rank 8 on 177 067 972 elements with Chow polynomial

$$\begin{aligned} \underline{H}(N; t) &= 1 + 963\,924\,470\,032\,260\,016\,646\,795\,095\,676t \\ &\quad + 772\,700\,986\,375\,783\,620\,699\,737\,483\,420\,137t^2 \\ &\quad + 668\,902\,642\,263\,664\,767\,416\,856\,175\,849\,115\,238t^3 \\ &\quad + 668\,902\,642\,263\,664\,767\,416\,856\,175\,849\,115\,238t^4 \\ &\quad + 772\,700\,986\,375\,783\,620\,699\,737\,483\,420\,137t^5 \\ &\quad + 963\,924\,470\,032\,260\,016\,646\,795\,095\,676t^6 + t^7. \end{aligned}$$

The coefficients a_1, a_2, a_3 of the terms t, t^2, t^3 satisfy

$$\frac{a_2^2}{a_1a_3} \approx 0.926013 < 1.$$

Hence, its Chow polynomial is not log-concave, and hence not real-rooted.

The projective family in rank 7 serves as counterexamples to the log-concavity of Chow polynomials. For every prime power $q \geq 512$, the matroid $C(q, q^3)$ is simple and 3-connected, and its Chow polynomial is not log-concave, and hence not real-rooted. More generally, the same conclusions hold for $C(q, \lfloor \rho q^3 \rfloor)$ for each fixed $0 < \rho \leq 1$ and all sufficiently large prime powers q .

Example 1.5 The matroid $C(512, 512^3)$ of rank 7 is simple and 3-connected, on

$$1 + 512 + 512^2 + 2 \cdot 512^3 + 512^4 + 512^5 = 35\,253\,360\,263\,681$$

elements. Its Chow polynomial is

$$\begin{aligned} \underline{H}(C; t) = & 1 + 12796287528694082360694375709107783381771733983561734t \\ & + 2260867910885166349838189869115390690568965414423633935t^2 \\ & + 402548512894232793390837832528792797279036205852996415508t^3 \\ & + 2260867910885166349838189869115390690568965414423633935t^4 \\ & + 12796287528694082360694375709107783381771733983561734t^5 + t^6. \end{aligned}$$

The coefficients a_1, a_2, a_3 of the terms t, t^2, t^3 satisfy

$$\frac{a_2^2}{a_1 a_3} \approx 0.992312 < 1.$$

Thus $\underline{H}(C; t)$ is not log-concave, and hence not real-rooted.

Remark 1.6 These families achieve the smallest possible ranks for failures of both log-concavity and real-rootedness of the chain polynomial and the Chow polynomial. Brändén and Leite proved that the chain polynomial and the chain polynomial of every matroid of rank at most 3 are real-rooted [BL25, Theorem 6.1]. For Chow polynomials, Coron, Ferroni and Li proved real-rootedness for every matroid of rank at most 6 [CFL26a, Corollary 8.5].

The nontrivial direct sums in Theorem A are disconnected and have large ground sets. Below, we obtain rank-7 counterexamples to the log-concavity of Chow polynomials directly from the rank-4 counterexamples to the log-concavity of the chain polynomials, by truncating the direct sum of two matroids in the latter family or taking their parallel connections. These constructions give counterexamples that are minimal in their ranks and have much smaller ground sets. These examples also distinguish failure of real-rootedness from failure of log-concavity. For example, the Chow polynomial of $\text{Tr}^7(M_q \oplus M_q)$ is log-concave but not real-rooted for odd prime powers $47 \leq q \leq 89$, and it becomes not log-concave for $q \geq 97$. Adding one coloop before truncation gives 3-connected examples with the same thresholds (Theorem C).

Theorem C *The matroids*

$$\text{Tr}^7(M_q \oplus M_q \oplus U_{1,1}), \quad \text{Tr}^7(G_{4,q} \oplus G_{4,q} \oplus U_{1,1})$$

are simple, 3-connected and of rank 7. The following hold:

- (a) For every odd prime power q , $\underline{H}(\text{Tr}^7(M_q \oplus M_q \oplus U_{1,1}); t)$ is not real-rooted exactly when $q \geq 47$, and is not log-concave exactly when $q \geq 97$.
- (b) For every prime power q , $\underline{H}(\text{Tr}^7(G_{4,q} \oplus G_{4,q} \oplus U_{1,1}); t)$ is not real-rooted exactly when $q \geq 37$, and is not log-concave exactly when $q \geq 81$.

Corollary D *Conjecture 1.1, Conjecture 1.2 and Conjecture 1.3 are false.*

Proof. The inequalities for the numbers of multichains in Theorem B, together with Theorem A, imply non-log-concavity of the chain, Chow, and augmented Chow polynomials. By Newton's inequalities, the three real-rootedness conjectures are false. $\square$

The key mechanism which allows us to transfer failures of log-concavity of multichain counts to other polynomial invariants, is given by Theorem A. We show that a

similar mechanism also applies to the study of the so-called *graph associahedra* and their h -polynomials.

Recall that for a connected graph G , its associated graph associahedron [CD04] is the simple polytope whose dual simplicial complex is the simplicial complex of tubings, which are compatible families of nonempty proper connected vertex sets. Its h -polynomial is denoted by $h_G(t) = \sum_j h_j(G)t^j$. We refer to [PRW08] for more details of graph associahedra and its connections to Boolean matroids with building sets. For several prominent families of graphs, these polynomials and their positivity properties have been studied under different names. For instance, the h -polynomials of complete graphs coincide with the Eulerian polynomials, for path graphs we get Narayana polynomials, for cycle graphs we obtain type B Narayana polynomials and finally, the h -polynomials of star graphs correspond to binomial-Eulerian polynomials. All these polynomials are known to be real-rooted using various techniques with wide range of connections to analysis, partial differential equations, topology and geometry (see [Fro10, pp. 829–830] for Eulerian polynomials, [Brä06, Corollary 7.2] for type A and type B Narayana polynomials, and [HZ19, Theorem 3.1] for binomial-Eulerian polynomials). However, we show in Section 5 that h -polynomials of graph associahedra are not necessarily log-concave, and hence not real-rooted.²

For a connected rooted graph (G, v) , let $\mathfrak{F}_v(G)$ be the poset of connected vertex sets containing v , ordered by inclusion. Let $Z_G(j) = Z_{\mathfrak{F}_v(G)}(j)$ count multichains $\{v\} = S_0 \subseteq \cdots \subseteq S_j = V(G)$ in $\mathfrak{F}_v(G)$ with j steps, and we write $W_s(G)$ (for “wedging”) for the graph obtained by identifying the roots of s disjoint copies of G . This construction will play the same role as direct sums in the above matroid setting. Indeed, we have the similar identities

$$\mathcal{L}(M^{\oplus s}) \cong \mathcal{L}(M)^s, \quad \mathfrak{F}_v(W_s(G)) \cong \mathfrak{F}_v(G)^s.$$

Theorem 2.13 and Theorem 2.14 imply an analogue of Theorem A for graphs: if a graph G has a non log-concave sequence of multichain counts, then the polynomials $h_{W_s(G)}(t)$ are not log-concave for any sufficiently large integer s . Thus, in both settings, a failure of log-concavity of the multichain counts produces a failure of log-concavity of the h -polynomial, after sufficiently many repetitions. We end by presenting a graph whose multichain counts are not log-concave.

Theorem E (Theorem 5.3) *There is a connected rooted graph G on 1540 vertices such that*

$$Z_G(37)^2 < Z_G(36)Z_G(38).$$

For every integer $s \geq 23$, the graph associahedron of $W_s(G)$ has an h -polynomial satisfying

$$h_{36}(W_s(G))^2 < h_{35}(W_s(G))h_{37}(W_s(G)).$$

In particular, this polynomial is not log-concave, and hence not real-rooted.

The graph G consists of a long path connected to a long periodic ladder graph via sparse bridges. Morally speaking, the competing contributions of the path graph and the ladder graphs in the multichain counts produces a failure of log-concavity of multichain counts. We present this construction, and conduct computer assisted

²The question of whether h -polynomials of graph associahedra was studied and raised to the fourth author SL by Andrew Sack.

computations and follow an analogous proof strategy of failure of log-concavity to the matroid case in Section 5.

AI disclosure. LLMs were substantially used in the searches of these counterexamples. In an upcoming update of this paper, we will release a detailed analysis on the search history from an AI-methodology point of view. A brief summary of the searches is as follows.

The third author CE found a non-constructive disproof of the real-rootedness conjecture for Chow polynomials using ChatGPT Astra on September 23, and communicated to the fourth author SL about the nature of the disproof. Upon suggestions by SL, Matt Larson, and Lorenzo Vecchi, disproofs of the real-rootedness of augmented Chow and chain polynomials were obtained similarly on September 29. These disproofs involved inexplicit matroids of rank at least 2^{65} , whose existence relies on probabilistic methods, with no claim to the failure of log-concavity.

Prompted by the lack of concrete and explicit counterexamples of low ranks, and the question of whether log-concavity can fail, on September 30, the second author BC found using ChatGPT Astra a counterexample to the log-concavity of h -polynomials of graph associahedra (a special case of built matroids) of very high vertex count. On October 4, the first author GB, BC, and SL then orchestrated and launched an LLM-search to target specifically the failure of log-concavity for both the chain polynomials and the Chow polynomials. About 5 hours and 12 hours later, SL found using Claude Opus 5.5 three explicit families of matroids as counterexamples to both the log-concavity of chain polynomials and Chow polynomials of lowest ranks possible, which are presented here; soon after, GB also found using ChatGPT Astra a different family of counterexamples to the log-concavity of Chow polynomials.

Generative LLMs were further used for drafting the reports for the initial findings, literature search, identifying common patterns among the examples such as hyperplane coalescence, and producing figures. The mathematical content was digested and reshaped from the LLM search reports, checked, and written by the authors.

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Organization. Section 2 reviews the polynomial invariants and proves Theorem A. Section 3 develops hyperplane coalescence and absorption, and studies the changes of the multichain counts under coalescence. Section 4 constructs the four geometric families, proves Theorem B in Section 4.4, and derives the 3-connected rank-7 counterexamples of Theorem C in Section 4.5.

2. POLYNOMIAL INVARIANTS AND MULTICHAIN COUNTS

In this section, we review basic properties of the relevant polynomial invariants of matroids, and prove Theorem A. We then develop the parallel amplification argument for graph associahedra under wedges. Throughout, M is a loopless matroid of positive rank r , $\mathcal{L}$ denotes the lattice of flats of M , and $\overline{\mathcal{L}}$ denotes the poset of nonempty proper flats of M .

2.1. Chain and face enumeration polynomials. Let $f_{k-1}(\mathcal{L})$ be the number of faces of dimension $k-1$ in the order complex of $\mathcal{L}$, with $f_{-1} = 1$ for the empty face. The face polynomials of $\mathcal{L}$ and $\overline{\mathcal{L}}$ are defined by

$$f(\mathcal{L}; t) = \sum_{k \geq 0} f_{k-1}(\mathcal{L}) t^k, \quad f(\overline{\mathcal{L}}; t) = \sum_{k=0}^{r-1} f_{k-1}(\overline{\mathcal{L}}) t^k.$$

For a matroid M , write $f(M; t) = f(\overline{\mathcal{L}}; t)$. For $S \subseteq [r-1] = \{1, \dots, r-1\}$, let $\alpha_M(S)$ count the chains in $\overline{\mathcal{L}}$ with rank set S . These numbers form the *flag f -vector*. The *flag h -vector* is defined by

$$\beta_M(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_M(T).$$

In particular, $W_M(i) = \alpha_M(i)$ is the number of rank- i flats, and $\alpha_M(\emptyset) = \beta_M(\emptyset) = 1$.

The order complex of $\overline{\mathcal{L}}$ has dimension $r-2$ and face numbers

$$f_{k-1}(\overline{\mathcal{L}}) = \sum_{\substack{S \subseteq [r-1] \\ |S|=k}} \alpha_M(S).$$

The h -polynomial of $\overline{\mathcal{L}}$ is defined by

$$h(M; t) := \sum_{k=0}^{r-1} f_{k-1}(\overline{\mathcal{L}}) t^k (1-t)^{r-1-k} = \sum_{S \subseteq [r-1]} \beta_M(S) t^{|S|}. \quad (1)$$

The flag h -numbers are nonnegative, and $\beta_{\mathcal{L}}([r-1]) = |\mu_{\mathcal{L}}(\widehat{0}, \widehat{1})|$ [Bjö92]. The chain and h -polynomials are related by

$$f(\mathcal{L}; t) = (1+t)^2 f(\overline{\mathcal{L}}; t), \quad f(\overline{\mathcal{L}}; t) = (1+t)^{r-1} h\left(M; \frac{t}{1+t}\right). \quad (2)$$

2.2. Multichain counts. We recall the background of the multichain counting polynomial introduced in [Sta12, Section 3.12] in the setting of matroids. For a loopless matroid M of positive rank r , the function Z_M counts multichains in its lattice of flats:

$$Z_M(m) := \#\{\widehat{0} = F_0 \leq F_1 \leq \dots \leq F_m = \widehat{1}\} \quad (m \geq 0).$$

In particular, $Z_M(0) = 0$ and $Z_M(1) = 1$. For $m \geq 2$, a multichain counted by $Z_M(m)$ is determined by its $m-1$ intermediate flats in $\mathcal{L}$, which may include $\widehat{0}$ and $\widehat{1}$. If these entries contain k distinct flats, they form a strict chain counted by $f_{k-1}(\mathcal{L})$. Their positive multiplicities sum to $m-1$, giving $\binom{m-2}{k-1}$ choices. Therefore,

$$Z_M(m) = \sum_{k=1}^{r+1} f_{k-1}(\mathcal{L}) \binom{m-2}{k-1} = \sum_{k=0}^{r-1} f_{k-1}(\overline{\mathcal{L}}) \binom{m}{k+1} \quad (m \geq 2), \quad (3)$$

with $Z_M(0) = 0$, $Z_M(1) = 1$, and $Z_M(m)$ is strictly increasing for integers $m \geq 1$. In particular,

$$\begin{aligned} f_0(\mathcal{L}) &= Z_M(2), \\ f_1(\mathcal{L}) &= Z_M(3) - Z_M(2), \\ f_2(\mathcal{L}) &= Z_M(4) - 2Z_M(3) + Z_M(2). \end{aligned}$$

Lemma 2.1 *For any matroid M ,*

$$f_1(\mathcal{L})^2 - f_0(\mathcal{L})f_2(\mathcal{L}) = Z_M(3)^2 - Z_M(2)Z_M(4). \quad (4)$$

Thus $f(\mathcal{L}; t)$ fails log-concavity at coefficient index 2 exactly when the values of the zeta polynomial fail log-concavity at $m = 3$.

One advantage of the zeta polynomial is its behavior over direct sums: a multichain in a product lattice is a tuple of multichains in its factors. By definition of multichain counts, we have

$$Z_{M^{\oplus n}}(m) = Z_M(m)^n. \quad (5)$$

Lemma 2.2 *If for $j \geq 2$,*

$$Z_M(j)^2 < Z_M(j-1)Z_M(j+1)$$

then the same inequality holds for $M^{\oplus n}$ for every $n \geq 1$. Furthermore, if $j = 3$, then $f(\mathcal{L}(M^{\oplus n}); t)$ fails to be log-concave at coefficient index 2.

Proof. The first statement follows from (5):

$$\frac{Z_{M^{\oplus n}}(j)^2}{Z_{M^{\oplus n}}(j-1)Z_{M^{\oplus n}}(j+1)} = \left(\frac{Z_M(j)^2}{Z_M(j-1)Z_M(j+1)} \right)^n < 1.$$

The second statement follows from (4). $\square$

2.3. Chow polynomials and augmented Chow polynomials. The Chow ring $\underline{\text{CH}}(M)$ and the augmented Chow ring $\text{CH}(M)$ have Hilbert–Poincaré polynomials

$$\underline{H}(M; t) = \sum_{i=0}^{r-1} \dim \underline{\text{CH}}^i(M) t^i, \quad H(M; t) = \sum_{i=0}^r \dim \text{CH}^i(M) t^i.$$

We first describe the Chow polynomial using chains of flats, using [FMSV24, Proposition 3.6] which is based on the Feichtner–Yuzvinsky Gröbner basis. For a nonempty subset $S = \{s_1 < \dots < s_k\} \neq \emptyset$, let

$$\begin{aligned} \underline{\tau}(S) &= t[s_1 - 1]_t \prod_{i=2}^k t[s_i - s_{i-1} - 1]_t, \\ \tau(S) &= t[s_1]_t \prod_{i=2}^k t[s_i - s_{i-1} - 1]_t, \end{aligned}$$

with $\underline{\tau}(\emptyset) = \tau(\emptyset) = 1$. Grouping chains by their proper-flat rank sets gives

$$\begin{aligned} \underline{H}(M; t) &= \sum_{S \subseteq [r-1]} \alpha_M(S) (\underline{\tau}(S) + \underline{\tau}(S \cup \{r\})), \\ H(M; t) &= \sum_{S \subseteq [r-1]} \alpha_M(S) (\tau(S) + \tau(S \cup \{r\})). \end{aligned} \quad (6)$$

The Chow and augmented Chow polynomials can be expressed using the flag h -vectors [Stu24, Theorems 1.1 and 1.2]. A set $S \subseteq [r-1]$ is *stable* if it contains no consecutive integers. Then

$$\begin{aligned}\underline{H}(M; t) &= \sum_{\substack{S \subseteq [r-1] \text{ stable} \\ 1 \notin S}} \beta_M(S) t^{|S|} (1+t)^{r-1-2|S|}, \\ H(M; t) &= \sum_{S \subseteq [r-1] \text{ stable}} \beta_M(S) t^{|S|} (1+t)^{r-2|S|}.\end{aligned}$$

2.4. Asymptotic coefficients. In this section, we give asymptotic formulas for the coefficients of the aforementioned polynomial invariants of the direct sum $M^{\oplus n}$ in terms of the multichain counts of M . First, we have the following explicit formula for every coefficient of the face enumeration polynomials of $M^{\oplus n}$.

Proposition 2.3 *For $n \geq 1$ and $j \geq 0$,*

$$\begin{aligned}f_{j-1}(M^{\oplus n}) &= \sum_{m=1}^{j+1} (-1)^{j+1-m} \binom{j+1}{m} Z_M(m)^n, \\ h_j(M^{\oplus n}) &= \sum_{m=1}^{j+1} (-1)^{j+1-m} \binom{nr+1}{j+1-m} Z_M(m)^n.\end{aligned}$$

Proof. The first identity follows by binomial inversion from Equation 3 and Equation 5. For the second identity, the identity

$$\sum_{m \geq 0} \binom{m}{k+1} t^m = \frac{t^{k+1}}{(1-t)^{k+2}}$$

implies that

$$\sum_{m \geq 1} Z_{M^{\oplus n}}(m) t^{m-1} = \sum_{k=0}^{nr-1} f_{k-1}(M^{\oplus n}) \frac{t^k}{(1-t)^{k+2}} = \frac{h(M^{\oplus n}; t)}{(1-t)^{nr+1}}.$$

Multiplying by $(1-t)^{nr+1}$, and substituting with $Z_{M^{\oplus n}}(m) = Z_M(m)^n$ gives the result. $\square$

For the asymptotic coefficients of the Chow and augmented Chow polynomials of $M^{\oplus n}$, we express the polynomials using incidence-algebraic techniques from the work of [FMV24]. The incidence algebra $\mathcal{F}(\mathcal{L})$ is an associative algebra, consisting of polynomials on closed intervals of $\mathcal{L}$. For a more general description of incidence algebra for posets, see [Sta12, Section 3.7]. In $\mathcal{F}(\mathcal{L})$, multiplication is convolution

$$(a \cdot b)(x, y; t) := \sum_{x \leq u \leq y} a(x, u; t) b(u, y; t), \text{ for every } x \leq y \text{ in } \mathcal{L},$$

and the multiplicative identity of $\mathcal{F}(\mathcal{L})$ is the delta function

$$\delta(x, y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

We also have the zeta function, together with the Möbius function as its inverse:

$$\zeta(x, y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}, \quad \mu(x, y) := \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq u < y} \mu(x, u) & \text{if } x < y \end{cases}.$$

For $\mathcal{L}$, $\mathcal{J}(\mathcal{L})$ admits an involution $a \mapsto a^{\text{rev}}$ defined by

$$a^{\text{rev}}(x, y; t) := t^{\text{rk}_M(y) - \text{rk}_M(x)} a(x, y; t^{-1}),$$

which reverses the coefficients of the polynomials $a(x, y; t)$ on an interval $[x, y]$. The *characteristic function* is defined as

$$\chi := \mu \cdot \zeta^{\text{rev}}, \text{ with } \zeta^{\text{rev}}(x, y; t) = t^{\text{rk}(y) - \text{rk}(x)},$$

and for $\mathcal{L}$, $\chi(\widehat{0}, \widehat{1})$ coincides with the characteristic polynomial of the matroid M . Following the construction and result in [FMV24, Definitions 3.2, 3.3, 3.13 and Theorems 4.9, 4.11], the Chow and left augmented Chow functions are defined as

$$\begin{aligned} \underline{H} &:= (1 - t)(\chi - t\delta)^{-1} = (1 - t) \sum_{m \geq 1} t^{m-1} \chi^{-m}, \\ H &:= \zeta^{\text{rev}} \cdot \underline{H} = (1 - t) \sum_{m \geq 1} t^{m-1} (\zeta^{\text{rev}} \cdot \chi^{-m}), \end{aligned} \tag{7}$$

with $\underline{H}(\widehat{0}, \widehat{1}; t) = \underline{H}(M; t)$ and $H(\widehat{0}, \widehat{1}; t) = H(M; t)$.

Proposition 2.4 *For every $n \geq 1$,*

$$\begin{aligned} \underline{H}(M^{\oplus n}; t) &= (1 - t) \sum_{m \geq 1} t^{m-1} ((\chi^{-m})(\widehat{0}, \widehat{1}; t))^n, \\ H(M^{\oplus n}; t) &= (1 - t) \sum_{m \geq 1} t^{m-1} ((\zeta^{\text{rev}} \cdot \chi^{-m})(\widehat{0}, \widehat{1}; t))^n. \end{aligned} \tag{8}$$

Proof. Since $\mathcal{L}(M^{\oplus n}) \cong \mathcal{L}(M)^n$, the functions ζ , ζ^{rev} , and χ in each summand of Equation 7 factor over the direct sum. Convolution and taking inverses respect this factorization, and the formulas then follow. $\square$

We have the following computational lemma relating the expressions above to multichain counts.

Lemma 2.5 *For any fixed $m \geq 1$, and a polynomial $Q_m(M; t)$ equal to*

$$(\chi_M^{-m})(\widehat{0}, \widehat{1}; t) \quad \text{or} \quad (\zeta_M^{\text{rev}} \cdot \chi_M^{-m})(\widehat{0}, \widehat{1}; t),$$

we have the constant terms

$$Q_m(M; 0) = \zeta_M^m(\widehat{0}, \widehat{1}) = Z_M(m), \quad Q_m(M^{\oplus n}; 0) = Z_M(m)^n.$$

Proof. We first consider the case of a single matroid M . For $(\chi_M^{-m})(\widehat{0}, \widehat{1}; t)$, evaluation at $t = 0$ commutes with convolution and therefore also with inversion. Since $\chi_M(0) = \mu_M$ and $\mu_M^{-1} = \zeta_M$, we obtain

$$\chi_M^{-m}(0) = \mu_M^{-m} = \zeta_M^m.$$

For $(\zeta_M^{\text{rev}} \cdot \chi_M^{-m})(\widehat{0}, \widehat{1}; t)$, since $\zeta_M^{\text{rev}}(0) = \delta_M$, which is the multiplicative identity, the constant term is unchanged:

$$(\zeta_M^{\text{rev}} \cdot \chi_M^{-m})(0) = \chi_M^{-m}(0) = \zeta_M^m.$$

Therefore, $Q_m(M; 0)$ is equal to

$$\zeta_M^m(\widehat{0}, \widehat{1}) = \sum_{\widehat{0} = F_0 \leq F_1 \leq \dots \leq F_m = \widehat{1}} \left(\prod_{i=1}^m \zeta_M(F_{i-1}, F_i) \right) = \sum_{\widehat{0} = F_0 \leq F_1 \leq \dots \leq F_m = \widehat{1}} 1 = Z_M(m).$$

We next apply the same argument to $M^{\oplus n}$. Since $\mathcal{L}(M^{\oplus n}) \cong \mathcal{L}(M)^n$, a multichain in this product is determined by an independent choice of a multichain in each coordinate. Therefore,

$$Q_m(M^{\oplus n}; 0) = Z_{M^{\oplus n}}(m) = Z_M(m)^n.$$

□

Proposition 2.6 *Let $P(M; t)$ denote any one of*

$$f(\mathcal{L}; t), \quad f(\overline{\mathcal{L}}; t), \quad h(M; t), \quad \underline{H}(M; t), \quad H(M; t),$$

and write $P(M; t) = \sum_j P_j(M) t^j$. For a fixed $j \geq 1$,

$$\lim_{n \rightarrow \infty} P_j(M^{\oplus n}) = Z_M(j+1)^n + O(n^j Z_M(j)^n). \quad (9)$$

Furthermore,

$$\lim_{n \rightarrow \infty} \frac{P_j(M^{\oplus n})}{Z_M(j+1)^n} = 1.$$

Proof. For $f(M; t)$ or $h(M; t)$, since $Z_M(m)$ is strictly increasing, Proposition 2.3 implies that as n goes to ∞ , the j th coefficient of $f(M^{\oplus n}; t)$, which is $f_{j-1}(M^{\oplus n})$, or that of $h(M^{\oplus n}; t)$, which is $h_j(M^{\oplus n})$, is dominated by the term $Z_M(j+1)^n$. Therefore, we have the asymptotic ratios

$$\lim_{n \rightarrow \infty} \frac{f_{j-1}(M^{\oplus n})}{Z_M(j+1)^n} = 1, \quad \lim_{n \rightarrow \infty} \frac{h_j(M^{\oplus n})}{Z_M(j+1)^n} = 1.$$

By (2), the same limit is true for the j th coefficient of $f(\mathcal{L}; t)$.

For $\underline{H}(M; t)$ and $H(M; t)$, if we write

$$Q_m(M^{\oplus n}; t) = \begin{cases} (\chi^{-m})(\widehat{0}, \widehat{1}; t) & \text{for } \underline{H} \\ (\zeta^{\text{rev}} \cdot \chi^{-m})(\widehat{0}, \widehat{1}; t) & \text{for } H \end{cases},$$

then Proposition 2.4 implies that

$$P(M^{\oplus n}; t) = \sum_{m \geq 1} (t^{m-1} - t^m) Q_m(M^{\oplus n}; t).$$

Therefore, for $j \geq 1$, the j th coefficient of $P(M^{\oplus n}; t)$ is

$$P_j(M^{\oplus n}) = Q_{j+1}(M; 0)^n + \sum_{m=1}^j (Q_{m, j+1-m}(M^{\oplus n}) - Q_{m, j-m}(M^{\oplus n})).$$

Then Lemma 2.5 and Lemma 2.7 imply that, as n goes to infinity, the first term $Q_{j+1}(M; 0)^n = Z_M(j+1)^n$ dominates, and

$$\lim_{n \rightarrow \infty} \frac{P_j(M^{\oplus n})}{Z_M(j+1)^n} = 1.$$

□

Lemma 2.7 *For either choice of Q_m , and for every fixed $j \geq 1$, there exists a constant $C = C_{M, j}$ such that, as $n \rightarrow \infty$,*

$$\sum_{m=1}^j (Q_{m, j+1-m}(M^{\oplus n}) - Q_{m, j-m}(M^{\oplus n})) \leq C_{M, j} n^j Z_M(j)^n.$$

Proof. By multiplicativity and Lemma 2.5,

$$Q_m(M^{\oplus n}; t) = Q_m(M; t)^n, \quad c_m := Q_m(M; 0) = Z_M(m) > 0.$$

Write $Q_m(M; t) = c_m(1 + tB_m(t))$, where $B_m(t)$ is independent of n . The binomial theorem implies

$$Q_{m,d}(M^{\oplus n}) = c_m^n \sum_{\ell=0}^{\min(n,d)} \binom{n}{\ell} B_{m,d-\ell}(t)^\ell,$$

where the coefficients $B_{m,d-\ell}(t)^\ell$ are independent of n . Since $0 \leq \ell \leq \min(n, d)$,

$$\binom{n}{\ell} \leq \frac{n^\ell}{\ell!} \leq \frac{n^d}{\ell!},$$

and there exists a constant $C_{M,m,d}$ such that

$$|Q_{m,d}(M^{\oplus n})| \leq C_{M,m,d} n^d Z_M(m)^n.$$

There are only finitely many pairs $1 \leq m \leq j$ and $0 \leq d \leq j$, so these bounds hold with a common constant $C = C(M, j)$. Therefore,

$$\left| \sum_{m=1}^j (Q_{m,j+1-m}(M^{\oplus n}) - Q_{m,j-m}(M^{\oplus n})) \right| \leq C \sum_{m=1}^j (n^{j+1-m} + n^{j-m}) Z_M(m)^n \leq 2jC n^j Z_M(j)^n.$$

Updating the bounding constant C gives the desired statement. $\square$

Remark 2.8 The work of [Pie25] gives a decomposition of Chow rings of direct sums of possibly different matroids using the Chow rings of the matroid summands. It would be interesting to obtain our asymptotic formula above using the decomposition.

Proof of Theorem A. Suppose that for any $j \geq 2$, $Z_M(j)^2 < Z_M(j-1)Z_M(j+1)$. By Proposition 2.6, for a fixed $j \geq 1$, as n goes to infinity, the j th coefficient $P_j(M^{\oplus n})$ is asymptotically equal to $Z_M(j+1)^n$. Furthermore,

$$P_0(M^{\oplus n}) = 1 = Z_M(1)^n.$$

Therefore, for any $j \geq 2$, the asymptotic ratio

$$\lim_{n \rightarrow \infty} \frac{P_{j-1}(M^{\oplus n})^2}{P_{j-2}(M^{\oplus n})P_j(M^{\oplus n})} = \lim_{n \rightarrow \infty} \left(\frac{Z_M(j)^2}{Z_M(j-1)Z_M(j+1)} \right)^n = 0 < 1,$$

which implies that log-concavity of coefficients of $P(M^{\oplus n})$ fails for sufficiently large n . Newton's inequalities in turn imply failures of real-rootedness [New07, Brä15]. $\square$

2.5. Face enumerations of graph associahedra. We now establish the graph analogue of Proposition 2.6. For matroids, direct sums give products of lattices of flats. For rooted graphs, wedge of graphs gives products of posets of connected vertex sets containing the root, which will play an analogous role.

The construction of a graph with non log-concave h -polynomial has two distinct steps. First, we find a rooted connected graph whose numbers of multichains of rooted connected sets fail log-concavity. Then identify the roots of sufficiently many copies of that graph. We prove that this operation transfers the failure of log-concavity of the multichain counts to the h -polynomial of the resulting graph associahedron, and we will construct the explicit rooted graph in Section 5.

2.5.1. *Tubings and height functions.* Throughout, we consider only finite simple graphs. For a finite simple graph G , let $\mathcal{B}(G)$ be the set of nonempty vertex sets inducing connected subgraphs. A family $\mathcal{S} \subseteq \mathcal{B}(G)$ is *nested* if every two members are comparable by inclusion, or disjoint with no edge between them. If G is a connected graph, a *tube* is a member of $\mathcal{B}(G) \setminus \{V(G)\}$, and a *tubing* is a nested family of tubes. The complex of tubings is the simplicial complex dual to the graph associahedron; it is pure of dimension $n - 2$, where n is the number of vertices of G . [PRW08]. Its h -polynomial is

$$h_G(t) = \sum_{\mathcal{T}} t^{|\mathcal{T}|} (1 - t)^{n-1-|\mathcal{T}|}, \quad (10)$$

where the sum runs over all tubings, including the empty tubing. We write $h_j(G) = [t^j]h_G(t)$. These coefficients are nonnegative. We use $h_{K_1}(t) = 1$ when a component consists of one vertex.

As recalled in Section 2.6, Newton's inequalities imply that a real-rooted polynomial with nonnegative coefficients is log-concave. Thus a strict reverse inequality for three consecutive h -coefficients also disproves real-rootedness.

For a height function $d : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, define

$$w_G(d) = \sum_{r \geq 1} \#\{\text{components of } G[\{x : d(x) \geq r\}]\}, \quad \Phi_G(t) = \sum_d t^{w_G(d)},$$

which is a well-defined power series, since $w_G(d) \geq \max d$, and only finitely many height functions contribute to any coefficient. For the empty graph, set $\Phi_{\emptyset}(t) = 1$. The following classical result shows that the h -polynomial of a connected graph G is the numerator of the generating series Φ_G . For completeness, we provide a proof.

Proposition 2.9 ([Zel06, Theorem 5.1]) *There is a weight-preserving bijection between height functions on G and nested families $\mathcal{S} \subseteq \mathcal{B}(G)$ equipped with positive integer multiplicities $\mu(T)$, under which*

$$d(x) = \sum_{T \ni x} \mu(T), \quad w_G(d) = \sum_{T \in \mathcal{S}} \mu(T).$$

Consequently

$$\Phi_G(t) = \sum_{\mathcal{S}} \left(\frac{t}{1-t} \right)^{|\mathcal{S}|} = \prod_C \frac{h_C(t)}{(1-t)^{|C|}}, \quad (11)$$

where C runs over the connected components of G . In particular, for connected G ,

$$h_G(t) = (1-t)^{|V(G)|} \Phi_G(t).$$

Proof. Given d , collect all components of all positive superlevel sets, and let $\mu(T)$ count the levels at which T occurs. Components at different levels are comparable or disjoint. Disjoint components cannot be adjacent: an edge would connect them at the lower level. Thus they form a nested family. Counting, for each vertex, the levels at which it occurs gives the formula for d , and counting components gives the formula for w_H .

Conversely, start with $(\mathcal{S}, \mu)$ and define d by the displayed formula. Put $D(T) = \sum_{U \supseteq T} \mu(U)$, with the sum over $U \in \mathcal{S}$. For $D(T) - \mu(T) < r \leq D(T)$, every vertex of T has height at least r . If a vertex outside T is adjacent to T , every member of $\mathcal{S}$ containing that vertex must properly contain T ; its height is therefore at most $D(T) - \mu(T)$. Hence T is a component of $\{d \geq r\}$. For each vertex x with $d(x) \geq r$,

the members of $\mathcal{S}$ containing x form a chain, and exactly one of their successive multiplicity intervals contains r . These components therefore cover the superlevel set, proving the bijection.

Summing the positive multiplicities gives the first equality in (11). Nested families split independently over components of H . For connected G , such a family is a tubing, possibly with $V(G)$ adjoined. Setting $y = t/(1 - t)$ therefore gives $\Phi_G(t) = (1 + y) \sum_{\mathcal{T}} y^{|\mathcal{T}|}$; now apply (10). $\square$

2.5.2. Rooted chains and products under gluing. Fix a connected rooted graph (G, v) with $n \geq 2$ vertices. Let $\mathcal{F}_v(G)$ be the poset, ordered by inclusion, of connected vertex sets containing v . For $j \geq 0$, put

$$Z_G(j) = \#\{\{v\} = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_j = V(G) : S_i \in \mathcal{F}_v(G)\},$$

which is the zeta polynomial $Z_{\mathcal{F}_v(G)}(j)$. In particular, we have

$$Z_G(0) = 0, \quad Z_G(1) = 1, \quad Z_G(2) = |\mathcal{F}_v(G)|.$$

For $j \geq 1$, a multichain counted by $Z_G(j)$ has j steps and $j - 1$ unrestricted intermediate sets. Throughout, a *rooted zeta witness at k* means

$$Z_G(k)^2 < Z_G(k - 1)Z_G(k + 1). \quad (12)$$

For $j \geq 1$, appending $V(G)$ to the list of intermediate sets injects the multichains counted by $Z_G(j)$ into those counted by $Z_G(j + 1)$. Its image misses the chain whose intermediate sets are all $\{v\}$. Together with $Z_G(0) = 0$, this proves strict increase. Moreover,

$$Z_G(3) \leq |\mathcal{F}_v(G)|^2 = Z_G(2)^2.$$

Thus the log-concavity inequalities at indices 1 and 2 hold, and a rooted zeta witness necessarily has $k \geq 3$.

If $G_1, \dots, G_s$ are rooted graphs, the *wedge* $G_1 \vee \cdots \vee G_s$ of rooted graphs is obtained by gluing the graphs $G_1, \dots, G_s$ along their respective roots. Write $W_q(G)$ for the wedge of q copies of G ; it has $1 + q(n - 1)$ vertices. We have the following simple lemma.

Lemma 2.10 (product under wedges) *Restriction to the copies induces a poset isomorphism*

$$\mathcal{F}_v(G_1 \vee \cdots \vee G_s) \cong \prod_{i=1}^s \mathcal{F}_{v_i}(G_i).$$

In particular, $Z_{W_q(G)}(j) = Z_G(j)^q$.

Proof. A vertex set containing the common root is connected exactly when its intersection with every copy is connected. The intersections recover the set by taking their union, and both operations preserve inclusion. A multichain in a product is a tuple of multichains. $\square$

Our next task is to relate the coefficients of the h -polynomial $h_G(t)$ to the multichain counts $Z_G(i)$. This will be achieved in the next section.

2.5.3. *The wedge formula.* Let $t_G = |\mathcal{B}(G - v)|$, and for $a \geq 0$ define

$$\Psi_a(t) = \sum_{d: d(v)=a} t^{w_G(d)-a}, \quad A_a(t) = (1-t)^{n-1} \Psi_a(t).$$

At levels $1, \dots, a$ there is a component containing v , so every exponent in Ψ_a is nonnegative. For a vertex set S , write $N[S]$ for its closed neighbourhood in G . Inequalities between series below mean coefficientwise inequalities.

Lemma 2.11 (rooted layers) *For each chain $\mathcal{C} : V(G) = S_0 \supseteq S_1 \supseteq \dots \supseteq S_a$ in $\mathcal{F}_v(G)$, put*

$$U_\ell = S_\ell \setminus N[S_{\ell+1}] \quad (0 \leq \ell < a), \quad U_a = S_a \setminus \{v\}, \quad U(\mathcal{C}) = \bigcup_{\ell=0}^a U_\ell.$$

Then

$$\Psi_a(t) = \sum_{\mathcal{C}} \Phi_{G[U(\mathcal{C})]}(t), \quad \Psi_a(0) = Z_G(a+1), \quad \Psi_a(t) \leq Z_G(a+1)(1-t)^{-t_G}. \quad (13)$$

Moreover $A_a(t)$ is a polynomial of degree at most $n-1$, with constant term $Z_G(a+1)$.

Proof. Use the nested family with multiplicities associated to a height function by Proposition 2.9. Its members containing v form a chain, and their multiplicities sum to $d(v) = a$. Listing these members in decreasing order, each repeated with its multiplicity, gives exactly $S_1, \dots, S_a$.

A connected set T avoiding v is compatible with this chain exactly when it lies in one of the sets U_ℓ . Indeed, take the largest ℓ for which $T \subseteq S_\ell$. If $\ell < a$, compatibility forces T to be disjoint from and nonadjacent to $S_{\ell+1}$, since T cannot contain the root-containing set $S_{\ell+1}$. If $\ell = a$, it lies in $S_a \setminus \{v\}$. The converse follows immediately from these same conditions. The U_ℓ are pairwise disjoint and nonadjacent: for $\ell < \ell'$, the set $U_{\ell'}$ lies in $S_{\ell+1}$, which has no vertex or neighbour in U_ℓ . Thus all the remaining weighted nested families are precisely those of $G[U(\mathcal{C})]$. Their weights contribute $w_G(d) - a$, proving the series identity. Reversing $S_1, \dots, S_a$ and adjoining the bottom $\{v\}$ and top $V(G)$ gives a multichain with $a+1$ steps. Thus there are $Z_G(a+1)$ chains, and each summand has constant term 1.

Every connected set of $G[U(\mathcal{C})]$ belongs to $\mathcal{B}(G - v)$. Forgetting the nestedness restriction bounds its weighted nested families by all multisets from this set of t_G possibilities. Their generating series is $(1-t)^{-t_G}$, proving the bound. Finally, for $U = U(\mathcal{C})$, Proposition 2.9 gives

$$(1-t)^{n-1} \Phi_{G[U]}(t) = (1-t)^{n-1-|U|} \prod_C h_C(t).$$

Since $U \subseteq V(G) \setminus \{v\}$, this is a polynomial, of degree at most

$$n-1-|U| + \sum_C (|C|-1) \leq n-1.$$

□

Theorem 2.12 (wedge formula) *For every integer $q \geq 1$,*

$$h_{W_q(G)}(t) = (1-t)^{1+q(n-1)} \sum_{a \geq 0} t^a \Psi_a(t)^q = (1-t) \sum_{a \geq 0} t^a A_a(t)^q. \quad (14)$$

These are identities of formal power series, whose left side is a polynomial.

Proof. Fix a height function on the wedge with common root height a , and let d_i be its restriction to copy i . At each of the first a levels, the q components containing the root merge into one. All other components remain separate. Hence

$$w_{W_q(G)}(d) = \sum_{i=1}^q w_G(d_i) - (q-1)a.$$

The d_i are independent once their root heights are fixed, so their total generating series is $t^a \Psi_a(t)^q$. Sum over a and apply the height-function formula to the wedge, which has $1 + q(n-1)$ vertices. The definition of A_a gives the second equality. $\square$

In particular, the contribution with $a = j$ to the coefficient of t^j is exactly $Z_G(j+1)^q$. Contributions with $a < j$ involve smaller numbers of multichains, and no term with $a > j$ contributes. The next estimate makes this separation effective without computing the polynomials A_a themselves.

2.5.4. *An explicit amplification certificate.* For $j, q \geq 1$, set

$$E_j(q) = \sum_{a=0}^{j-1} \left(\frac{Z_G(a+1)}{Z_G(j+1)} \right)^q \binom{q(n-1+t_G) + j - a}{j - a}, \quad E_0(q) = 0. \quad (15)$$

Theorem 2.13 (coefficient estimate) *For every $j, q \geq 1$,*

$$\left| \frac{h_j(W_q(G))}{Z_G(j+1)^q} - 1 \right| \leq E_j(q). \quad (16)$$

For fixed G and j , this gives

$$h_j(W_q(G)) = Z_G(j+1)^q + O(q^j Z_G(j)^q) \quad (q \rightarrow \infty).$$

Proof. Put $D = 1 + q(n-1)$. Extracting the coefficient in (14) yields

$$h_j(W_q(G)) = \sum_{a=0}^j [t^{j-a}] (1-t)^D \Psi_a(t)^q.$$

The term $a = j$ is $Z_G(j+1)^q$. For $m = j - a \geq 1$, the absolute coefficients of $(1-t)^D$ are those of $(1+t)^D$, and $(1+t)^D \leq (1-t)^{-D}$ coefficientwise. Since Ψ_a has nonnegative coefficients, Lemma 2.11 gives

$$\begin{aligned} |[t^m](1-t)^D \Psi_a(t)^q| &\leq Z_G(a+1)^q [t^m](1-t)^{-D-qt_G} \\ &= Z_G(a+1)^q \binom{q(n-1+t_G) + m}{m}. \end{aligned}$$

Summing and dividing by $Z_G(j+1)^q$ proves the bound. For fixed j , each binomial factor is a polynomial in q of degree at most j , and $Z_G(a+1) \leq Z_G(j)$ for $a < j$, proving the asymptotic statement. $\square$

Theorem 2.14 (graph zeta amplification) *Suppose (G, v) is a rooted zeta witness at k , and put*

$$\rho = \frac{Z_G(k)^2}{Z_G(k-1)Z_G(k+1)} < 1.$$

If an integer $q \geq 1$ satisfies $E_{k-2}(q) < 1$, $E_k(q) < 1$, and

$$\rho^q \frac{(1 + E_{k-1}(q))^2}{(1 - E_{k-2}(q))(1 - E_k(q))} < 1, \quad (17)$$

then

$$h_{k-1}(W_q(G))^2 < h_{k-2}(W_q(G))h_k(W_q(G)).$$

All these conditions hold for every sufficiently large q . As a consequence, the h -polynomials of these wedges are not log-concave, and hence not real-rooted.

Proof. The coefficient estimate places $h_j(W_q(G))$ between $Z_G(j+1)^q(1 - E_j(q))$ and $Z_G(j+1)^q(1 + E_j(q))$. The assumed positive lower bounds at $k-2$ and k , together with nonnegativity at $k-1$, therefore give

$$\frac{h_{k-1}(W_q(G))^2}{h_{k-2}(W_q(G))h_k(W_q(G))} \leq \rho^q \frac{(1 + E_{k-1}(q))^2}{(1 - E_{k-2}(q))(1 - E_k(q))} < 1.$$

Strict increase of Z_G implies $Z_G(a+1)/Z_G(j+1) < 1$ whenever $a < j$. Each summand of $E_j(q)$ is thus a fixed polynomial in q times a decaying exponential, so $E_j(q) \rightarrow 0$. Also $\rho^q \rightarrow 0$, proving eventual validity of the certificate. Newton's inequalities give the failure of real-rootedness. $\square$

In Section 5 we will construct a rooted graph G which satisfies the hypothesis above. The above theorems will allow us to produce an example of a wedge of G having a non log-concave h -polynomial. For this, it will be enough to know the multichain counts $Z_G(1), \dots, Z_G(k+1)$, the number t_G of connected sets avoiding the root, and an integer q passing (17). No enumeration of all tubings of the much larger graph $W_q(G)$ is needed.

2.6. Log-concavity and real-rootedness. We briefly recall relevant properties of polynomials, and refer the reader to Brändén's survey [Brä15]. A polynomial $p(t) = \sum_{i=0}^d a_i t^i \in \mathbb{R}_{\geq 0}[t]$ is *log-concave*, if $a_i^2 \geq a_{i-1}a_{i+1}$ for every i . For every i , the *defect at index i* is defined to be $a_i^2 - a_{i-1}a_{i+1}$. If $p(t)$ only has real zeros, the well-known result of Newton [New07, pp. 242–245] implies *ultra log-concavity* of the coefficients of $p(t)$:

$$\frac{a_i^2}{\binom{d}{i}^2} \geq \frac{a_{i-1}}{\binom{d}{i-1}} \frac{a_{i+1}}{\binom{d}{i+1}}, \quad \text{for every } 0 < i < d,$$

which implies log-concavity with no internal zeros.

If a polynomial $p(t) = \sum_{i=0}^d a_i t^i \in \mathbb{R}_{\geq 0}[t]$ is palindromic, it admits a unique expansion into the basis $\{t^i(1+t)^{d-2i}\}_{i \geq 0}$, and has γ -polynomial

$$p(t) = \sum_i \gamma_i t^i (1+t)^{d-2i}, \quad \gamma(p(t); x) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i.$$

Since the Chow ring of a matroid is a Poincaré duality algebra, the Chow polynomial is palindromic. By [Stu24, FMV24], its γ -coefficients can be written using the flag h -vector as follows:

$$\underline{H}(\mathbf{M}; t) = \sum_i \gamma_i t^i (1+t)^{r-1-2i}, \quad \gamma_i = \sum_{\substack{S \subseteq [r-1] \text{ stable} \\ 1 \notin S, |S|=i}} \beta_{\mathbf{M}}(S). \quad (18)$$

We will be particularly interested in Chow polynomials of degree 6. For any matroid of rank 7, the only stable subset of $\{2, \dots, 6\}$ with size 3 is $\{2, 4, 6\}$. Therefore,

$$\gamma_0 = 1, \quad \gamma_3 = \beta_M(246),$$

and the coefficients of the Chow polynomial satisfy

$$\begin{aligned} \underline{H}_0 &= \underline{H}_6 = 1, \\ \underline{H}_1 &= \underline{H}_5 = \gamma_1 + \binom{6}{1}, \\ \underline{H}_2 &= \underline{H}_4 = \gamma_2 + \binom{4}{1}\gamma_1 + \binom{6}{2}, \\ \underline{H}_3 &= \gamma_3 + \binom{2}{1}\gamma_2 + \binom{4}{2}\gamma_1 + \binom{6}{3}. \end{aligned} \tag{19}$$

Lemma 2.15 *Let M be a loopless matroid of rank 7.*

(a) *$\underline{H}(M; t)$ is log-concave if and only if*

$$\Delta_{\underline{H}} := \underline{H}_2^2 - \underline{H}_1 \underline{H}_3 \geq 0. \tag{20}$$

(b) *$\underline{H}(M; t)$ is real-rooted if and only if the discriminant for the cubic polynomial $\gamma(\underline{H}(t); x)$ is nonnegative:*

$$\text{disc}(\gamma(\underline{H}(t); x)) = \gamma_1^2 \gamma_2^2 - 4\gamma_1^3 \gamma_3 - 4\gamma_2^3 + 18\gamma_1 \gamma_2 \gamma_3 - 27\gamma_3^2 \geq 0.$$

Proof. The first statement follows from the definition and palindromicity of $\underline{H}(M; t)$. The second statement follows from the equivalence of real-rootedness of a polynomial and its γ -polynomial. See, for instance, [Pet15, Observation 4.2]. $\square$

3. MATROID OPERATIONS: HYPERPLANE COALESCENCE, ABSORPTION

We study the properties of two crucial matroid operations, which are essential in our construction: hyperplane coalescence, and a special class of extensions by modular cuts, called *absorption*. For background on matroids, we refer to [Oxl11].

We start with the classical theory of detruncation, introduced by Crapo [Cra70] to construct matroids with prescribed hyperplanes and a fixed truncation. See also [PvdP17, Section 3 and Theorem 12].

Definition 3.1 Let M be a matroid of positive rank, and $k \geq 0$. The k th truncation $T^k(M)$ of M is the matroid with rank $r - k$ with lattice of flats

$$\mathcal{L}(T^k(M)) = \{F \in \mathcal{L}(M) : \text{rk}_M(F) < k\} \cup \{E\}.$$

If $k = 1$, the truncation $T^k(M)$ is called the *principal truncation*, or simply *truncation*. A matroid M of rank r is a *detruncation* of a matroid T of rank $r - 1$, if the principal truncation $\text{Tr}(M)$ is T .

The following properties are standard.

Lemma 3.2 *For a matroid M of rank r on E and $1 \leq k \leq r$, the following hold.*

(a) *The flats of $\text{Tr}^k(M)$ are the flats of M of rank at most $k - 1$, with their rank in M , and E . In particular $\alpha_{\mathcal{L}(\text{Tr}^k(M))}(S) = \alpha_{\mathcal{L}(M)}(S)$ for every $S \subseteq [k - 1]$ ([Oxl11, Proposition 7.3.10]).*

- (b) If M is loopless and $k < r$, then $\text{Tr}^k(M)$ is connected ([Oxl11, Section 7.3, Exercise 8]).
- (c) If M is simple and $k \geq 2$, then $\text{Tr}^k(M)$ is simple ([Oxl11, Proposition 7.3.10]).

3.1. Hyperplane coalescence. Hyperplane coalescence is a natural geometric operation on matroids that merges groups of hyperplanes into larger hyperplanes while preserving the geometry of lower ranks. It is an application of Crapo's criterion in the Crapo–Knuth theory of detruncations [Cra70, Knu75, PvdP17]. The spanning subsets that become new hyperplanes are called *blocks*; the lemma below specifies which blocks may be selected. For a single block with restriction $U_{r,|H|}$, coalescence replaces this restriction by $U_{r-1,|H|}$, reversing stressed-hyperplane relaxation [FNV22, Theorem 3.4 and Proposition 3.10]. When $|H| = r$, this is tightening a free basis.

Definition 3.3 (Hyperplane coalescence) Let M be a matroid of rank $r \geq 2$ on E , and let $\mathcal{H}$ be a family of proper spanning subsets of E . A matroid M' of rank r on E is obtained from M via *hyperplane coalescence along $\mathcal{H}$* if:

- (a) M and M' have the same flats of rank at most $r - 2$, with the same ranks;
- (b) the hyperplanes of M' are precisely the elements in $\mathcal{H}$ together with the hyperplanes of M not contained in any element in $\mathcal{H}$.

In other words, the collection of hyperplanes of M' is

$$\mathcal{L}^{r-1}(M') = \mathcal{H} \cup \{F \in \mathcal{L}^{r-1}(M) : F \not\subseteq H \text{ for every } H \in \mathcal{H}\}.$$

The members of $\mathcal{H}$ are called *blocks*. We say that M' is obtained by *coalescing the blocks $\mathcal{H}$ in M* . Thus, to *coalesce* a block H is to replace the hyperplanes contained in H by the single hyperplane H . For each block H , the hyperplanes in M contained in H are replaced by the single hyperplane H in M' .

Example 3.4 For $U_{3,4}$, the coalescence along $\mathcal{H} = \{123\}$ of $U_{3,4}$ is $U_{2,3} \oplus U_{1,1}$. See Figure 1.

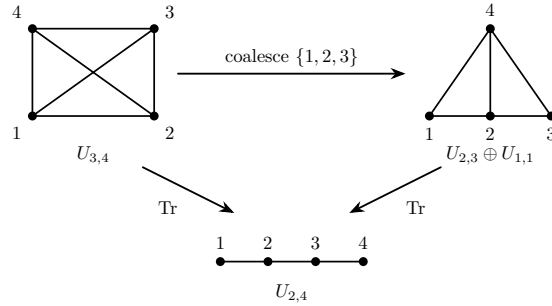


FIGURE 1. A coalescence of $U_{3,4}$. Both $U_{3,4}$ and its coalescence have the same truncation.

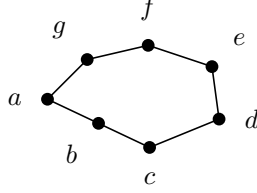
The following example shows that hyperplane coalescence need not always exist for an arbitrary collection $\mathcal{H}$ of proper spanning subsets of E .

Example 3.5 Let $M = U_{3,4}$ on $E = [4]$, and consider

$$\mathcal{H} = \{123, 124\}.$$

Both members of $\mathcal{H}$ are proper subsets of full rank 3. However, hyperplane coalescence along $\mathcal{H}$ does not exist. Indeed, any such matroid M' would be simple, since its flats of rank 1 are the same as those of M . The two blocks would then be distinct flats of rank 2 containing the independent pair $\{1, 2\}$. Both would therefore equal $\text{cl}_{M'}(\{1, 2\})$, a contradiction.

Example 3.6 In Crapo's original work on detruncations, there is an example with all 13 detruncations of the following combinatorial geometry of rank 3.



The *free detruncation* N of a matroid M of rank r is a matroid of rank $r + 1$ such that for every detruncation N' of M , the independent sets of N contain those of N' :

$$\mathcal{I}(N') \subseteq \mathcal{I}(N).$$

Figure 2 shows all the detruncations of Crapo's geometry of rank 3, with its free detruncation on top. All the detruncations can be viewed as hyperplane coalescences of the free detruncation along a set of blocks $\mathcal{H}$.

The following lemma gives a sufficient condition on $\mathcal{H}$ for a coalescence to exist, and it shows that when such a matroid exists, it is unique.

Lemma 3.7 *Let M have rank $r \geq 2$ on E , and let $\mathcal{H}$ be a family of proper spanning subsets satisfying*

- (a) $\text{rk}_M(H \cap H') \leq r - 2$ for distinct $H, H' \in \mathcal{H}$;
- (b) $\text{cl}_M(I) \subseteq H$ whenever $I \subseteq H \in \mathcal{H}$ is independent and $|I| = r - 1$.

Then $\mathcal{H}$ uniquely determines a hyperplane coalescence M' of M , which has bases

$$\mathcal{B}(M') = \{B \in \mathcal{B}(M) : B \not\subseteq H \in \mathcal{H}\}.$$

and rank function

$$\text{rk}_{M'}(S) = \begin{cases} r - 1, & \text{rk}_M(S) = r \text{ and } S \subseteq H \text{ for some } H \in \mathcal{H}, \\ \text{rk}_M(S), & \text{otherwise.} \end{cases} \quad (21)$$

Proof. This follows from Crapo's criterion [PvdP17, Theorem 12] applied to $\text{Tr}(M)$. $\square$

One advantage of the hyperplane coalescence is that it changes multichain counts in a controlled way, enabling us to apply Theorem A. Since only the hyperplanes change under coalescence, the change in the zeta polynomial is a sum of contributions from the individual blocks. Let M be loopless of rank $r \geq 2$, let $\mathcal{H}$ be a collection of blocks satisfying Lemma 3.7, and let M' be its coalescence along $\mathcal{H}$.

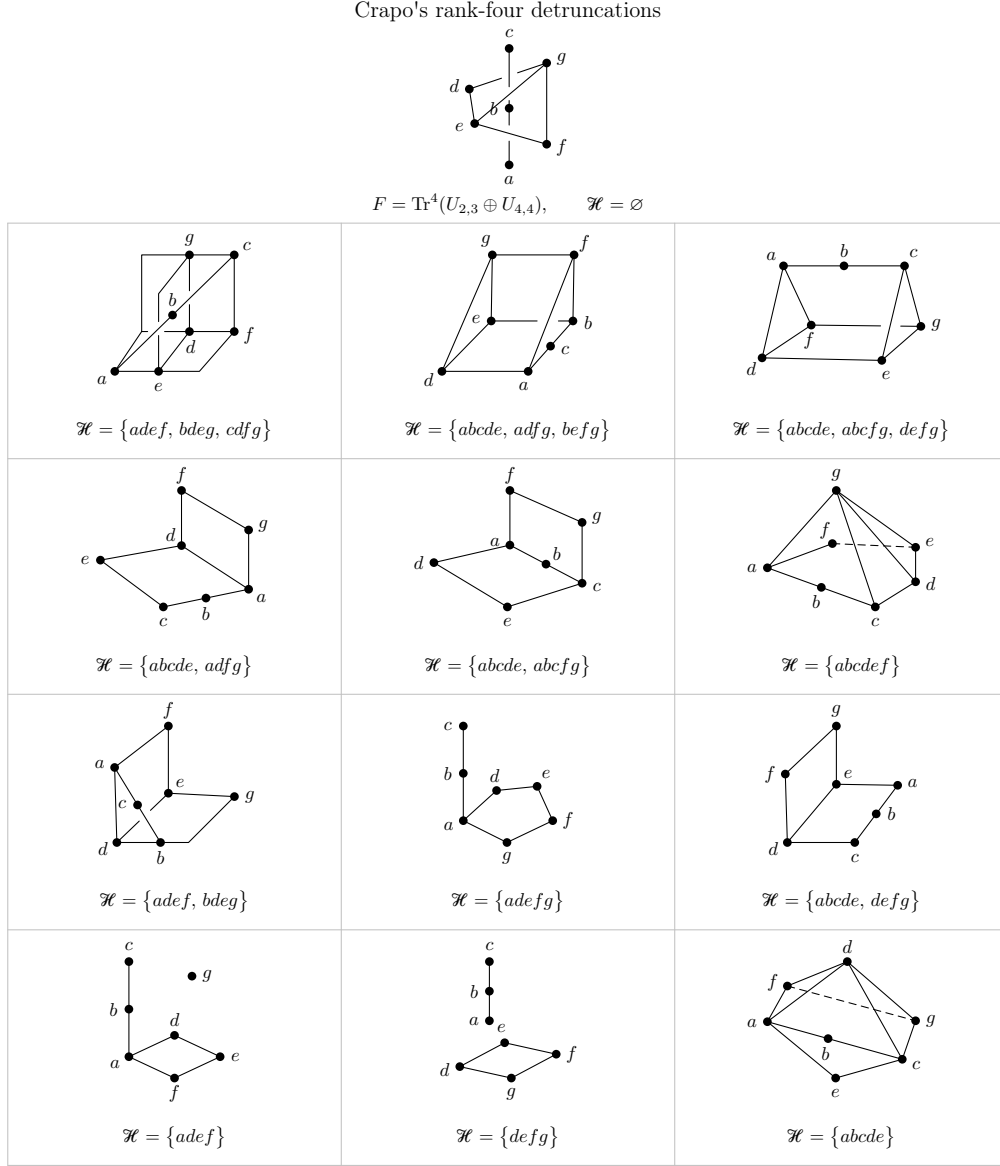


FIGURE 2. The rank-4 detruncations of Crapo's geometry

Proposition 3.8 *For every $m \geq 0$,*

$$Z_{M'}(m) = Z_M(m) - \sum_{H \in \mathcal{H}} \left(Z_{M^H}(m) - \sum_{i=0}^m Z_{\text{Tr}(M^H)}(i) \right). \quad (22)$$

This will let us compute the multichain counts of our geometric families without enumerating their lattices. To prove the proposition, we count the affected flags.

Lemma 3.9 *For every $S \subseteq [r-2]$,*

$$\begin{aligned} \alpha_{M'}(S) &= \alpha_M(S), \\ \alpha_{M'}(S \cup \{r-1\}) &= \alpha_M(S \cup \{r-1\}) + \sum_{H \in \mathcal{H}} (\alpha_{M^H}(S) - \alpha_{M^H}(S \cup \{r-1\})). \end{aligned} \quad (23)$$

Proof. This follows from the fact that flats of rank less than $r - 1$ are the same for M and M' , together with the closure hypothesis in Lemma 3.7. $\square$

Lemma 3.10 *For $P = f$ or $P = \underline{H}$,*

$$P(M'; t) = P(M; t) - \sum_{H \in \mathcal{H}} (P(M^H; t) - (1 + t)P(\text{Tr}(M^H); t)). \quad (24)$$

Proof. Summing (23) by chain length gives the assertion for f . For the Chow polynomial, the weights in (6) satisfy, for $S \subseteq [r - 2]$,

$$\tau(S \cup \{r\}) = t(\tau(S) + \tau(S \cup \{r - 1\})), \quad \tau(S \cup \{r - 1, r\}) = 0.$$

Thus (6) implies that, for any loopless matroid M of rank r ,

$$\underline{H}(M; t) - (1 + t)\underline{H}(\text{Tr}(M); t) = \sum_{S \subseteq [r-2]} (\alpha_M(S \cup \{r - 1\}) - \alpha_M(S))\tau(S \cup \{r - 1\}).$$

Applying (23) with these weights proves the result. $\square$

Proof of Proposition 3.8. The binomial transform (3) is a linear map which sends t^j to $\binom{t}{j+1}$, and the image of $(1 + t)t^j$ under the binomial transform is a polynomial, whose evaluation at m is

$$\binom{m}{j+1} + \binom{m}{j+2} = \binom{m+1}{j+2} = \sum_{a=0}^m \binom{a}{j+1}.$$

The assertion follows by applying this transform to (24) with $P = f$. $\square$

3.2. Absorption. In this subsection, we use hyperplane coalescence to construct extensions of a matroid by adjoining auxiliary points, each lying in the closure of every hyperplane in a prescribed family. We call this construction *absorption*. We first adjoin the auxiliary points as coloops and truncate back to the original rank. The union of each selected hyperplane with its assigned point is then a spanning subset. Under the conditions below, these subsets can be coalesced simultaneously, placing each auxiliary point in the required closures. We use this construction for the affine and Gabidulin families.

Proposition 3.11 *Let G have rank $r \geq 2$, and let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be pairwise disjoint families of hyperplanes such that*

$$\text{rk}_G(F \cap F') \leq r - 3 \quad (F, F' \in \mathcal{F}_i, F \neq F').$$

For a set of new points $X = \{p_1, \dots, p_m\}$, put

$$M = \text{Tr}^r(G \oplus U_{X,X}), \quad \mathcal{H} = \{F \cup \{p_i\} : F \in \mathcal{F}_i, 1 \leq i \leq m\}.$$

Then the blocks in $\mathcal{H}$ can be coalesced in M . The resulting matroid M' restricts to G and, for $A \subseteq E(G)$ and $J \subseteq X$, has rank

$$\text{rk}_{M'}(A \cup J) = \begin{cases} r - 1, & J = \{p_i\} \text{ and } \text{cl}_G(A) \in \mathcal{F}_i, \\ \min\{\text{rk}_G(A) + |J|, r\}, & \text{otherwise.} \end{cases} \quad (25)$$

Before coalescence, each block has restriction

$$M|(F \cup \{p_i\}) = (G|F) \oplus U_{1,1}.$$

For each $F \in \mathcal{F}_i$, the old hyperplanes F and $A \cup \{p_i\}$, with A a hyperplane of $G|F$, all acquire closure $F \cup \{p_i\}$. The rank formula also shows that the order of adjoining the auxiliary points does not affect the result.

Proof. Put $Q = G \oplus U_{X,X}$. Each block $H = F \cup \{p_i\}$ is a proper rank- r flat of Q , hence a proper spanning subset of M . For every independent $(r-1)$ -set $I \subseteq H$,

$$\text{cl}_M(I) = \text{cl}_Q(I) \subseteq H.$$

Distinct blocks sharing p_i intersect in rank at most $(r-3) + 1 = r-2$. Blocks with different auxiliary points intersect in rank at most $r-2$, since their geometric hyperplanes are distinct. Thus both conditions of Lemma 3.7 hold.

A set $A \cup J$ is spanning in M and contained in a block $F \cup \{p_i\}$ exactly when $J = \{p_i\}$ and $\text{cl}_G(A) = F$. Hence (21) gives (25); taking $J = \emptyset$ shows that M' restricts to G . The old block restriction follows because truncation to rank r does not change the restriction to a rank- r subset. $\square$

4. FOUR FAMILIES FROM FINITE GEOMETRIES

We construct the affine Klein family, the affine packing family, the Gabidulin family, and the projective family from finite geometries, and prove Theorem B.

4.1. The affine family. Let q be a power of an odd prime. Recall that the affine geometry $\text{AG}(4, q)$ is the matroid of rank 5, which has flats given by affine subspaces of $\mathbb{F}_q^4$, and the projective geometry $\text{PG}(3, q)$ is the matroid of rank 4, which has flats given by projective subspaces of $\mathbb{P}(\mathbb{F}_q^4)$. We will construct the affine family of matroids M_q using classical operations from projective incidence geometry and combinatorial design, in four steps:

- (i) Start with the truncation $\text{Tr}(\text{AG}(4, q))$.
- (ii) Use the Klein correspondence to choose pairwise disjoint spreads of $\text{PG}(3, q)$. Each spread specifies a family of affine planes through their directions.
- (iii) Construct the matroid inductively: for each chosen spread, adjoin a new coloop to the current matroid, truncate once, and absorb the new point into the selected planes. Define M_q as the resulting matroid.
- (iv) Compute flag vectors of M_q , and prove that they only depend on q .

For any field $\mathbb{F}$ and a vector space V over $\mathbb{F}$, the *direction* of a nonempty affine subspace $A \subseteq V$ is the linear subspace of V

$$D(A) := \{x - y : x, y \in A\} \subseteq V.$$

For example, the direction $D(L)$ is equal to L for any linear subspace $L \subseteq V$; and the direction of any affine translation $v + L$ of a linear space L for $v \in V$ is L . The projectivization of the direction of an affine subspace of dimension k is a projective subspace of dimension $k-1$. Therefore, for each flat F of rank k in $\text{AG}(n, q)$, $\mathbb{P}(D(F))$ is a flat of rank $k-1$ in $\text{PG}(n-1, q)$.

A k -spread of $\text{PG}(n, q)$ is a collection $\mathcal{S}$ of projective k -dimensional subspaces such that every projective point belongs to exactly one element of $\mathcal{S}$. A 1-spread is called a *line spread*, or simply *spread*. Two k -spreads are *disjoint* if they contain no common points. A k -spread of $\text{PG}(n, q)$ exists if and only if $k+1$ divides $n+1$, and every

such spread has cardinality

$$|\mathcal{S}| = \frac{q^{n+1} - 1}{q^{k+1} - 1}.$$

For $\text{PG}(3, q)$, every spread $\mathcal{S}$ consists of $q^2 + 1$ projective lines. For general background on spreads in finite projective spaces, see [Dem68, Hir98, Etz15].

Given a spread $\mathcal{S}$, we consider the collection of all affine planes whose projectivized directions are elements in $\mathcal{S}$

$$\mathcal{A}(\mathcal{S}) := \{A \subseteq V : \mathbb{P}(D(A)) \in \mathcal{S}\}.$$

Since each direction in $\mathcal{S}$ admits q^2 affine translates,

$$|\mathcal{A}(\mathcal{S})| = q^2(q^2 + 1).$$

The following lemma describes the intersections among the elements in $\mathcal{A}(\mathcal{S})$.

Lemma 4.1 *Let V be a vector space over a field, and let $P = a + D$ and $P' = b + D'$ be distinct affine subspaces of V .*

- (a) *If $D = D'$, then $P \cap P' = \emptyset$.*
- (b) *If $D \cap D' = \{0\}$, then $P \cap P'$ contains at most one point.*
- (c) *If $V = D \oplus D'$, then $P \cap P'$ contains exactly one point.*

In particular, if $V = \mathbb{F}_q^{2k+1}$ and $\mathbb{P}(D), \mathbb{P}(D')$ belong to a k -spread of $\text{PG}(2k, q)$, then the distinct affine k -dimensional subspaces P and P' are either parallel and disjoint, or have complementary directions and intersect in exactly one point.

Proof. Since a, b are distinct vectors, (a) follows immediately. For (b), if $x, y \in P \cap P'$, then $x - y \in D \cap D' = \{0\}$, and $x = y$. For (c), write $b - a = d + d'$ with $d \in D$ and $d' \in D'$. Then $a + d = b - d'$ belongs to $P \cap P'$, and (b) implies its uniqueness. $\square$

Example 4.2 Let $V = \mathbb{F}_q^4$ and let $\mathcal{S}$ be a spread of $\text{PG}(3, q)$. Let two distinct affine subspaces $P = a + D, P' = b + D'$ of V be given by distinct directions $\mathbb{P}(D) \neq \mathbb{P}(D')$ in $\mathcal{S}$. Since elements in $\mathcal{S}$ do not share common points, Lemma 4.1 implies that two distinct affine planes P, P' in $\mathcal{A}(\mathcal{S})$ are either parallel and disjoint, or meet in exactly one point. See Figure 3.

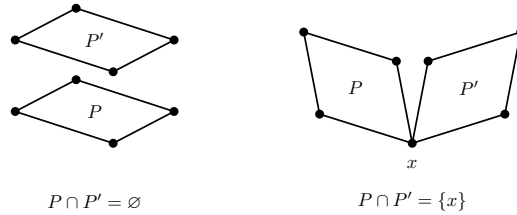


FIGURE 3.

Next, we use classical Plücker embeddings of Grassmannians to construct $\frac{q^2+1}{2}$ pairwise disjoint spreads of $\text{PG}(3, q)$. The Plücker embedding realizes the Grassmannian $\text{Gr}(2, V)$ of two-dimensional linear subspaces of an n -dimensional vector space V as a subvariety of $\mathbb{P}^{\binom{n}{2}-1} \cong \mathbb{P}(\bigwedge^2 V)$:

$$\iota: \text{Gr}(2, V) \hookrightarrow \mathbb{P}\left(\bigwedge^2 V\right), \quad \text{span}(u, v) \mapsto [u \wedge v].$$

When $n = 4$, the image is defined by a single quadratic equation, given by the Plücker relations. This special case is classically known as the *Klein correspondence*, and its image as the *Klein quadric*, studied by Klein in his 1868 dissertation, republished in [Kle84]. Over finite fields, the Klein correspondence identifies regular spreads with elliptic sections of the Klein quadric. For usage of Klein correspondence to construct regular spreads in the existing literature, see [Hir85, Section 15.4 and Chapter 17]. Let $V = \mathbb{F}_q^4$ with basis $v_1, \dots, v_4$, and denote

$$v_{ij} := v_i \wedge v_j \in \bigwedge^2 V, \quad W := \bigwedge^2 V$$

The Plücker relation defines the quadratic form on W with its bilinear form

$$\begin{aligned} Q: W &\rightarrow \mathbb{F}_q, & x &\mapsto x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} \\ B: W^2 &\rightarrow \mathbb{F}_q, & (x, y) &\mapsto Q(x + y) - Q(x) - Q(y). \end{aligned}$$

The *Klein quadric* is the image of the Plücker embedding, or equivalently, the vanishing locus of the quadratic form Q

$$\mathcal{K} := \{ [x] \in \mathbb{P}(W) : Q(x) = 0 \},$$

and the Plücker embedding gives the set bijection

$$\kappa : \mathcal{L}_2(\text{PG}(3, q)) \longrightarrow \mathcal{K}, \quad \mathbb{P}(\text{span}(u, v)) \longmapsto [u \wedge v].$$

In addition, we also have the set bijection by definition,

$$\gamma : \mathcal{L}_2(\text{PG}(3, q)) \xrightarrow{\sim} \text{Gr}(2, V).$$

Therefore, there is a commutative diagram of set morphisms

$$\begin{array}{ccc} \mathcal{L}_2(\text{PG}(3, q)) & \xrightarrow{\gamma} & \text{Gr}(2, V) \\ & \searrow \kappa & \swarrow \iota \\ & \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5 & \end{array} \quad (26)$$

A quadric in $\mathbb{P}^3$ is *elliptic* if it is smooth, and contains no projective lines. An *elliptic section* of the Klein quadric $\mathcal{K}$ is an intersection with a $\mathbb{P}^3$ which contains no projective lines. Under the Klein correspondence, elliptic sections correspond precisely to regular spreads of $\text{PG}(3, q)$. Therefore, our task reduces to constructing $\frac{q^2+1}{2}$ pairwise disjoint elliptic sections of $\mathcal{K}$.

Lemma 4.3 *The Klein correspondence*

$$\kappa : \mathcal{L}_2(\text{PG}(3, q)) \longrightarrow \mathcal{K}$$

induces a bijection between elliptic sections of $\mathcal{K}$ and regular spreads of $\text{PG}(3, q)$, given by

$$\mathcal{E} \longmapsto \mathcal{S}(\mathcal{E}) := \kappa^{-1}(\mathcal{E}) \subseteq \mathcal{L}_2(\text{PG}(3, q)).$$

Proof. Let $\mathcal{E} = \mathcal{K} \cap \mathbb{P}(U)$ be an elliptic section. If two distinct lines in $\mathcal{S}(\mathcal{E})$ intersected, the projective line joining their Klein images would lie entirely on $\mathcal{K}$. It would also lie in $\mathbb{P}(U)$, hence in $\mathcal{E}$, contradicting ellipticity. Thus the projective lines

of $\mathcal{S}(\mathcal{E})$ are pairwise skew. An elliptic quadric has $q^2 + 1$ points, so $\mathcal{S}(\mathcal{E})$ consists of $q^2 + 1$ pairwise skew lines. Since each projective line contains $q + 1$ points, and

$$(q^2 + 1)(q + 1) = q^3 + q^2 + q + 1 = |\mathrm{PG}(3, q)|,$$

these lines partition the point set of $\mathrm{PG}(3, q)$ and form a spread. The fact that this spread is regular, and that every regular spread arises from an elliptic section, is the classical regular-spread correspondence [Hir85, Section 15.4 and Chapter 17]. Furthermore, since κ is a bijection, the regular spread from an elliptic section is unique. $\square$

To construct $\frac{q^2+1}{2}$ pairwise disjoint elliptic sections of $\mathcal{K}$, we use *polarity* given by the nondegenerate bilinear form B on $\mathcal{K}$. A projective line $\ell \subseteq \mathbb{P}(W)$ is *external* to $\mathcal{K}$, if $\ell \cap \mathcal{K} = \emptyset$. The quadratic form Q has associated nondegenerate symmetric bilinear form

$$B(x, y) = Q(x + y) - Q(x) - Q(y).$$

For a linear subspace $A \subseteq W$, we define its orthogonal complement by

$$A^{\perp_B} := \{w \in W : B(w, a) = 0 \text{ for every } a \in A\}.$$

The *polar* of the projective subspace $\mathbb{P}(A)$ is the projectivization of this complement

$$\mathbb{P}(A)^{\perp} := \mathbb{P}(A^{\perp_B}).$$

Since W has dimension $\binom{4}{2} = 6$, the polar of a projective line is a $\mathbb{P}^3$. Moreover, if $\langle S, T \rangle$ denotes the projective span of two projective subspaces, then

$$(S^{\perp})^{\perp} = S, \quad \langle S, T \rangle^{\perp} = S^{\perp} \cap T^{\perp}.$$

For the Klein quadric $\mathcal{K}$, the polar of an external line meets $\mathcal{K}$ in an elliptic section. Thus every external line ℓ determines an elliptic quadric $\mathcal{K} \cap \ell^{\perp}$ and hence a regular spread of $\mathrm{PG}(3, q)$. We will construct several external lines so that their polar projective spaces meet only outside $\mathcal{K}$.

Construction 4.4 Let q be an odd prime power. Choose an external line

$$\ell_0 = \mathbb{P}(A_0) \subseteq \mathbb{P}(W) \quad \text{with } \dim A_0 = 2,$$

and define the orthogonal complement and its associated elliptic section of $\mathcal{K}$,

$$U = A_0^{\perp_B}, \quad \mathcal{E}_0 = \mathcal{K} \cap \mathbb{P}(U).$$

Then $\dim U = 4$, $\mathbb{P}(U) = \ell_0^{\perp}$, and $\mathcal{E}_0$ is an elliptic quadric. By Lemma 4.6, there exists a fixed-point-free projective involution τ of $\mathbb{P}(U)$ satisfying

$$\tau^2 = \mathrm{id}, \quad \tau(x) \neq x \text{ for every } x \in \mathbb{P}(U), \quad \tau(\mathcal{E}_0) = \mathcal{E}_0.$$

Let $\{\ell_1, \dots, \ell_m\}$ denote the distinct lines spanned by $\{x, \tau(x)\}$ for $x \in \mathbb{P}(U)$ which are disjoint from $\mathcal{E}_0$. These lines are external to $\mathcal{K}$, since they lie in $\mathbb{P}(U)$ and are disjoint from $\mathcal{E}_0 = \mathcal{K} \cap \mathbb{P}(U)$. We therefore define for each $1 \leq i \leq m$,

$$\mathcal{E}_i := \mathcal{K} \cap \ell_i^{\perp}, \quad \mathcal{S}_i := \kappa^{-1}(\mathcal{E}_i).$$

Lemma 4.5 Construction 4.4 gives $\frac{q^2+1}{2}$ pairwise disjoint elliptic sections of $\mathcal{K}$.

Proof. Each line $\text{span}(x, \tau(x))$ is preserved by τ . Two distinct such lines cannot meet, since their unique intersection point would be fixed by τ . Every point lies on its joining line, so these lines form a spread of $\mathbb{P}(U)$, consisting of $q^2 + 1$ lines. Each of these lines meets $\mathcal{E}_0$ in either zero or two points. Indeed, a line containing $x \in \mathcal{E}_0$ also contains the distinct point $\tau(x) \in \mathcal{E}_0$; it cannot contain more than two points of the quadric without lying entirely on it. Since $\mathcal{E}_0$ has $q^2 + 1$ points, exactly half the spread lines meet it, and the other half avoid it. Therefore $m = \frac{q^2+1}{2}$.

For $i \neq j$, the lines ℓ_i, ℓ_j are skew and hence $\text{span } \mathbb{P}(U)$. Therefore,

$$\ell_i^\perp \cap \ell_j^\perp = \langle \ell_i, \ell_j \rangle^\perp = \mathbb{P}(U)^\perp = \ell_0.$$

Their polar spaces meet precisely along the external line ℓ_0 . Hence

$$\mathcal{E}_i \cap \mathcal{E}_j = \mathcal{K} \cap \ell_0 = \emptyset.$$

The corresponding regular spreads $\mathcal{S}_i, \mathcal{S}_j$ therefore share no common elements. $\square$

Lemma 4.6 *For an odd prime power q , the projective 3-space $\mathbb{P}(U)$ in Construction 4.4 admits a fixed-point-free projective involution preserving $\mathcal{E}_0$.*

Proof. In diagonal coordinates, we write $Q = Q_1 \oplus Q_2$, where

$$Q_i(s, t) = a_i s^2 + b_i t^2, \quad a_i, b_i \neq 0.$$

Fix a nonsquare $d \in \mathbb{F}_q$. For each i , the sets of values u^2 and $d - (a_i/b_i)v^2$ each have $(q+1)/2$ elements, so they intersect. Thus we can choose u_i, v_i satisfying

$$u_i^2 + \frac{a_i}{b_i} v_i^2 = d.$$

The matrices

$$J_i = \begin{pmatrix} u_i & v_i \\ (a_i/b_i)v_i & -u_i \end{pmatrix}$$

satisfy $J_i^2 = dI$ and $Q_i(J_i x) = dQ_i(x)$. Therefore, $J = J_1 \oplus J_2$ satisfies

$$J^2 = dI, \quad Q(Jx) = dQ(x).$$

Its projectivization $\tau([x]) = [Jx]$ is an involution preserving $\mathcal{E}_0$. A fixed point would give $Jx = \lambda x$ for some $x \neq 0$ and $\lambda \in \mathbb{F}_q$ with $\lambda^2 = d$, a contradiction. $\square$

We now associate a matroid with any collection of pairwise disjoint spreads. At each step, we adjoin one coloop, truncate once, and perform the prescribed absorption.

Definition 4.7 Let q be a prime power, let $V = \mathbb{F}_q^4$, and let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be pairwise disjoint spreads of $\text{PG}(3, q)$. Choose distinct new elements $p_1, \dots, p_m$ outside V . Define matroids $M_0, \dots, M_m$ inductively as follows. Set

$$M_0 := \text{Tr}^1(\text{AG}(4, q)).$$

For $1 \leq i \leq m$, let

$$\tilde{M}_i := \text{Tr}^1(M_{i-1} \oplus U_{\{p_i\}, \{p_i\}}), \quad \mathcal{H}_i := \{P \cup \{p_i\} : P \in \mathcal{A}(\mathcal{S}_i)\}.$$

Let M_i be the hyperplane coalescence of $\tilde{M}_i$ along $\mathcal{H}_i$, and let

$$M(\mathcal{S}_1, \dots, \mathcal{S}_m) = M_m.$$

Definition 4.8 (The affine Klein family) For odd q , let $\{\mathcal{S}_1, \dots, \mathcal{S}_{(q^2+1)/2}\}$ be the collection of spreads given by Construction 4.4. The *affine Klein family* is defined by

$$M_q := M(\mathcal{S}_1, \dots, \mathcal{S}_{(q^2+1)/2}).$$

Proposition 4.9 Let q be a prime power, let $V = \mathbb{F}_q^4$, and let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be pairwise disjoint spreads of $\text{PG}(3, q)$. Let

$$X = \{p_1, \dots, p_m\}, \quad E = V \sqcup X, \quad \mathcal{A}_i = \mathcal{A}(\mathcal{S}_i).$$

Then $M(\mathcal{S}_1, \dots, \mathcal{S}_m)$ is a simple matroid of rank 4 on $q^4 + m$ elements, with proper nonempty flats given by

- (1) rank 1: the singletons of E ;
- (2) rank 2: the affine lines and the two-element subsets of E meeting X ;
- (3) rank 3: the affine planes outside $\bigcup_i \mathcal{A}_i$, the sets $P \cup \{p_i\}$ with $P \in \mathcal{A}_i$, and the three-element subsets of E containing at least two elements of X .

In particular, M_q is a simple matroid of rank 4 on $q^4 + \frac{q^2+1}{2}$ elements.

Proof. We induct on the number of absorptions, starting from $M_0 = \text{Tr}^1(\text{AG}(4, q))$. Since the spreads are pairwise disjoint, the planes in $\mathcal{A}_i$ remain hyperplanes of M_{i-1} . Since members of $\mathcal{A}_i$ intersect in rank at most 1, Proposition 3.11 implies that the resulting matroid is M_i . Adjoining p_i as a coloop and truncating once retains the old hyperplanes and creates the hyperplanes $L \cup \{p_i\}$ for L a flat of rank 2 of M_{i-1} . Since every affine line lies in exactly one plane of $\mathcal{A}_i$, coalescence replaces each selected plane P by $P \cup \{p_i\}$ and removes precisely the hyperplanes $L \cup \{p_i\}$ with L an affine line. The other new hyperplanes are exactly the listed triples containing p_i . The new flats of ranks 1 and 2 in the intermediate matroid are $\{p_i\}$ and $\{z, p_i\}$ for $z \in E(M_{i-1})$. Since hyperplane coalescence preserves all flats of these ranks, this establishes the inductive step. $\square$

For an odd prime power q , a *packing*, introduced by [Den72, Den73], is a partition of all lines of $\text{PG}(3, q)$ into $q^2 + q + 1$ spreads.

Definition 4.10 (The affine packing family) For an odd prime power q , let $\{\mathcal{S}_1, \dots, \mathcal{S}_{q^2+q+1}\}$ be a packing. The *affine packing family* is defined by

$$M_q^{\text{pk}} := M(\mathcal{S}_1, \dots, \mathcal{S}_{q^2+q+1}).$$

The matroid M_q^{pk} is a simple matroid of rank 4 on $q^4 + q^2 + q + 1$ elements.

4.1.1. *The flag f -vector of an affine family.* We compute the chain and h -polynomials by first counting flats and then incidences between them. Fix pairwise disjoint spreads $\mathcal{S}_1, \dots, \mathcal{S}_m$ of $\text{PG}(3, q)$, and write $M = M(\mathcal{S}_1, \dots, \mathcal{S}_m)$. Let N_1, N_2, N_3 denote the numbers of points, affine lines, and affine planes in $V = \mathbb{F}_q^4$, respectively. Then

$$\begin{aligned} N_1 &= q^4, \\ N_2 &= q^3(q^3 + q^2 + q + 1), \\ N_3 &= q^2(q^2 + 1)(q^2 + q + 1). \end{aligned}$$

Indeed, there are $q^3 + q^2 + q + 1$ line directions and $(q^2 + 1)(q^2 + q + 1)$ plane directions, with q^3 and q^2 affine translates per direction, respectively. Each spread contains $q^2 + 1$

plane directions, so

$$|\mathcal{A}(\mathcal{S}_i)| = q^2(q^2 + 1).$$

Since the spreads are pairwise disjoint, their selected affine planes are distinct.

Let W_i denote the number of flats of rank i , or *the i th Whitney number* of M . The description of flats of M in Proposition 4.9 implies the following formula for W_i :

$$\begin{aligned} W_1 &= N_1 + m, \\ W_2 &= N_2 + N_1 m + \binom{m}{2}, \\ W_3 &= N_3 + N_1 \binom{m}{2} + \binom{m}{3}. \end{aligned}$$

Corollary 4.11 *The flag f -vector of M is*

$$\begin{aligned} \alpha(i) &= W_i, \\ \alpha(12) &= qN_2 + 2N_1 m + 2\binom{m}{2}, \\ \alpha(13) &= q^2 N_3 + m q^2 (q^2 + 1) + 3N_1 \binom{m}{2} + 3\binom{m}{3}, \\ \alpha(23) &= (q^2 + q)N_3 + m q^4 (q^2 + 1) + 3N_1 \binom{m}{2} + 3\binom{m}{3}, \\ \alpha(123) &= q(q^2 + q)N_3 + 2m q^4 (q^2 + 1) + 6N_1 \binom{m}{2} + 6\binom{m}{3}. \end{aligned}$$

In particular, the flag f -vector of M depends only on q and m ; the choices $m = (q^2 + 1)/2$ and $m = q^2 + q + 1$ give the flag f -vectors of M_q for odd prime powers q and M_q^{pk} for all prime powers q , respectively.

Proof. It is immediate that $\alpha(\emptyset) = 1$, and $\alpha(i) = W_i$. There are N_2 affine lines, each containing q points, and $N_1 m + \binom{m}{2}$ two-point lines containing an auxiliary point. Counting points on these lines gives

$$\alpha(12) = qN_2 + 2 \left(N_1 m + \binom{m}{2} \right).$$

For the other entries, we group chains by their largest flat:

- (1) There are $N_3 - m q^2 (q^2 + 1)$ unchanged affine planes, each containing q^2 points and $q^2 + q$ lines, and contributing q^2 , $q^2 + q$, and $q(q^2 + q)$ to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.
- (2) There are $m q^2 (q^2 + 1)$ enlarged planes $P \cup \{p_i\}$, each containing $q^2 + 1$ points and $2q^2 + q$ lines: the $q^2 + q$ affine lines of P , each of size q , and the q^2 two-point lines $\{x, p_i\}$ with $x \in P$, and contributing $q^2 + 1$, $2q^2 + q$, and $q(q^2 + q) + 2q^2$ to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.
- (3) There are $N_1 \binom{m}{2} + \binom{m}{3}$ three-element hyperplanes, each containing three points and three lines, all of size 2, and contributing 3, 3, and 6 to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.

Summing these contributions over all hyperplanes gives the stated formulas. $\square$

4.2. The Gabidulin family. We first describe absorption in a truncated projective geometry. A Gabidulin code will then provide the families of subspaces needed for this construction.

Fix a prime power q and integers $n \geq k \geq 3$, and let $V = \mathbb{F}_q^n$. For a linear subspace $X \subseteq V$, write $\mathbb{P}(X)$ for its set of one-dimensional subspaces, with $\mathbb{P}(0) = \emptyset$. The matroid

$$M_0 = \text{Tr}^{n-k}(\text{PG}(n-1, q))$$

has rank k and ground set $\mathbb{P}(V)$. Here Tr^0 is the identity. Its proper flats are the sets $\mathbb{P}(X)$ with $\dim X \leq k-1$, of matroid rank $\dim X$.

Let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be pairwise disjoint families of $(k-1)$ -dimensional linear subspaces of V , satisfying

$$\dim(X \cap X') \leq k-3 \quad \text{for distinct } X, X' \in \mathcal{S}_i. \quad (27)$$

The corresponding hyperplanes $\mathbb{P}(X)$ are called *geometric blocks*. We next adjoin one point for each family and absorb it into every geometric block of that family.

Definition 4.12 Let $X_0 = \{p_1, \dots, p_m\}$ be a set of distinct auxiliary points disjoint from $\mathbb{P}(V)$. Starting from M_0 , define M_i for $1 \leq i \leq m$ as follows. First adjoin p_i as a coloop and truncate once:

$$\tilde{M}_i = \text{Tr}^1(M_{i-1} \oplus U_{\{p_i\}, \{p_i\}}).$$

Then obtain M_i from $\tilde{M}_i$ by coalescing the blocks

$$\mathcal{H}_i = \{\mathbb{P}(X) \cup \{p_i\} : X \in \mathcal{S}_i\}.$$

The resulting matroid is M_m .

Proposition 4.13 *The construction in Definition 4.12 is well-defined. The proper flats of M_m are the following, where $X \subseteq V$ is a linear subspace and $T \subseteq X_0$:*

- (L) *The flats $\mathbb{P}(X) \cup T$ with $\dim X + |T| \leq k-2$, of rank $\dim X + |T|$.*
- (N) *The unselected geometric hyperplanes $\mathbb{P}(X)$ with $\dim X = k-1$ and $X \notin \mathcal{S}_1 \cup \dots \cup \mathcal{S}_m$.*
- (B) *The coalesced hyperplanes $\mathbb{P}(X) \cup \{p_i\}$ with $X \in \mathcal{S}_i$.*
- (U) *The unabsorbed hyperplanes $\mathbb{P}(X) \cup \{p_i\}$ with $\dim X = k-2$ and X contained in no member of $\mathcal{S}_i$.*
- (H) *The hyperplanes $\mathbb{P}(X) \cup T$ with $|T| \geq 2$ and $\dim X + |T| = k-1$.*

Proof. We induct on the number of absorption steps. The assertion holds for M_0 . Suppose it holds for M_{i-1} . Since the families are pairwise disjoint, every $\mathbb{P}(X)$ with $X \in \mathcal{S}_i$ remains a hyperplane of M_{i-1} . The intersection bound (27) therefore allows absorption by Proposition 3.11.

The hyperplanes of $\tilde{M}_i$ are the same hyperplanes as those of M_{i-1} and the sets $F \cup \{p_i\}$, where F is a rank- $(k-2)$ flat of M_{i-1} . Hyperplane coalescence replaces each selected $\mathbb{P}(X)$ by $\mathbb{P}(X) \cup \{p_i\}$. Among the new hyperplanes, it removes precisely those $\mathbb{P}(Y) \cup \{p_i\}$ for which Y is contained in a member of $\mathcal{S}_i$. Hyperplanes containing an earlier auxiliary point remain, since they cannot lie in a selected block.

Adjoining p_i and truncating results in the lower flats listed in (L), and these are unchanged by coalescence. The remaining and coalesced hyperplanes give (N), (B), (U), and (H), completing the induction. $\square$

We now compute the flag f -vector when $k = 4$. Let N_1, N_2, N_3 denote the numbers of points, lines, and planes in $\text{PG}(n-1, q)$:

$$N_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \quad N_2 = \begin{bmatrix} n \\ 2 \end{bmatrix}_q, \quad N_3 = \begin{bmatrix} n \\ 3 \end{bmatrix}_q,$$

with $\begin{bmatrix} n \\ i \end{bmatrix}_q$ denoting the number of i -dimensional subspaces of $\mathbb{F}_q^n$. Each projective line contains $q+1$ points, and each projective plane contains $[3]_q = q^2 + q + 1$ points and the same number of lines. The pairwise disjoint families $\mathcal{S}_1, \dots, \mathcal{S}_m$ select a total of

$$b = \sum_{i=1}^m |\mathcal{S}_i|$$

geometric planes. Hyperplane coalescence replaces each selected plane $\mathbb{P}(X)$, with $X \in \mathcal{S}_i$, by $\mathbb{P}(X) \cup \{p_i\}$. Thus there are $N_3 - b$ unchanged geometric planes and b coalesced planes. For each auxiliary point p_i , the other possible hyperplanes containing no further auxiliary point are $\ell \cup \{p_i\}$, where ℓ is a projective line. Such a hyperplane is absorbed precisely when ℓ lies in a plane selected by $\mathcal{S}_i$. Each selected plane contains $[3]_q$ lines, and (27) ensures that no line lies in two planes selected by the same family. Therefore, the number of unabsorbed hyperplanes of this form is

$$N_U = mN_2 - b[3]_q.$$

The remaining hyperplanes contain at least two auxiliary points. By Proposition 4.13, they are the triples consisting of one geometric point and two auxiliary points, or three auxiliary points. As a result, the number of such hyperplanes is

$$N_1 \binom{m}{2} + \binom{m}{3}.$$

Let W_j denote the number of rank- j flats of M_m . The description of the flats above and the counts give

$$\begin{aligned} W_1 &= N_1 + m, \\ W_2 &= N_2 + N_1 m + \binom{m}{2}, \\ W_3 &= N_3 + N_U + N_1 \binom{m}{2} + \binom{m}{3}. \end{aligned}$$

Corollary 4.14 *The flag f -vector of M_m is*

$$\begin{aligned} \alpha(i) &= W_i, \\ \alpha(12) &= (q+1)N_2 + 2N_1 m + 2 \binom{m}{2}, \\ \alpha(13) &= [3]_q N_3 + b + (q+2)N_U + 3N_1 \binom{m}{2} + 3 \binom{m}{3}, \\ \alpha(23) &= [3]_q N_3 + [3]_q b + (q+2)N_U + 3N_1 \binom{m}{2} + 3 \binom{m}{3}, \\ \alpha(123) &= (q+1)[3]_q N_3 + 2[3]_q b + 3(q+1)N_U + 6N_1 \binom{m}{2} + 6 \binom{m}{3}. \end{aligned}$$

Proof. There are N_2 geometric lines, each containing $q + 1$ points, and $N_1m + \binom{m}{2}$ two-point lines involving auxiliary points. Counting points on these lines gives

$$\alpha(12) = (q + 1)N_2 + 2 \left(N_1m + \binom{m}{2} \right).$$

For the other entries, group chains by their largest flat.

- (1) There are $N_3 - b$ unselected geometric planes. Each contains $[3]_q$ points and $[3]_q$ lines, all of size $q + 1$. It therefore contributes $[3]_q$, $[3]_q$, and $(q + 1)[3]_q$ to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.
- (2) There are b coalesced planes. Each contains $[3]_q + 1$ points and $2[3]_q$ lines: the $[3]_q$ geometric lines, each of size $q + 1$, and the $[3]_q$ two-point lines joining a geometric point to the auxiliary point. It therefore contributes $[3]_q + 1$, $2[3]_q$, and $(q + 3)[3]_q$ to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.
- (3) There are N_U unabsorbed hyperplanes with one auxiliary point. Each contains $q + 2$ points and $q + 2$ lines: one geometric line of size $q + 1$, and $q + 1$ two-point lines through the auxiliary point. It therefore contributes $q + 2$, $q + 2$, and $3(q + 1)$ to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.
- (4) There are $N_1 \binom{m}{2} + \binom{m}{3}$ three-element hyperplanes. Each contains three points and three lines, all of size 2. It therefore contributes 3, 3, and 6 to $\alpha(13)$, $\alpha(23)$, and $\alpha(123)$, respectively.

Summing these contributions over all hyperplanes gives the stated formulas. $\square$

4.3. The projective family. Let q be a prime power and $4 \leq N \leq q^3$. We construct a rank-7 matroid $\mathcal{C}(q, N)$ by hyperplane coalescence, then compute its flag numbers and zeta values. Let $P_d(q) = \text{PG}(d - 1, q)$, with $P_0(q)$ empty. Let

$$U = \mathbb{F}_q^3, \quad K = \mathbb{F}_{q^3}, \quad V = U \oplus K.$$

Let X be a disjoint copy of $\mathbb{Z}/N\mathbb{Z}$, and put

$$Q = P_6(q) \oplus U_{X,X}, \quad M = \text{Tr}^{N-1}(Q).$$

Thus M has rank 7. Its proper flats are $\mathbb{P}(W) \sqcup D$, where $W \subseteq V$ is a linear subspace, $D \subseteq X$, and $\dim W + |D| \leq 6$.

Choose an $\mathbb{F}_q$ -basis $\alpha_1, \alpha_2, \alpha_3$ of K . For the standard basis e_1, e_2, e_3 of U , define

$$\phi : \text{Hom}_{\mathbb{F}_q}(U, K) \longrightarrow K, \quad A \longmapsto \sum_{i=1}^3 \alpha_i A(e_i).$$

Each map A determines a three-dimensional graph

$$G_A = \{(u, A(u)) : u \in U\}.$$

The map ϕ is surjective, so each fiber has q^6 elements. Moreover, its kernel contains no rank-one map: if $A(e_i) = c_i v$, with $v \neq 0$ and $c_i \in \mathbb{F}_q$, then

$$\phi(A) = 0 \implies \sum_i c_i \alpha_i = 0 \implies A = 0.$$

Consequently, distinct maps in the same fiber satisfy

$$\dim(G_A \cap G_{A'}) = 3 - \text{rk}(A - A') \leq 1.$$

Definition 4.15 Fix an injection $\iota : X \rightarrow K$. For each four-element subset $B \subseteq X$, let $\sigma(B) = \sum_{b \in B} b$, computed in $\mathbb{Z}/N\mathbb{Z}$. Define

$$\mathcal{H} = \{\mathbb{P}(G_A) \sqcup B : |B| = 4, \phi(A) = \iota(\sigma(B))\}.$$

With these choices fixed, the *projective family* $\mathcal{C}(q, N)$ is obtained by coalescing $\mathcal{H}$ in M . There are

$$c := |\mathcal{H}| = q^6 \binom{N}{4}$$

selected blocks.

To verify that coalescence is possible, each block is a proper rank-7 flat of Q , hence a spanning subset of M . Distinct blocks meet in rank at most 5: equal graphs force equal subset sums, so their distinct auxiliary sets share at most two elements; equal auxiliary sets give graph intersections of dimension at most one; otherwise the two contributions are at most two and three. For an independent six-element subset I of a block H ,

$$\text{cl}_M(I) = \text{cl}_Q(I) \subseteq H.$$

These are precisely the two conditions of Lemma 3.7.

The resulting matroid is simple of rank 7. If $N \geq 13$, it is 3-connected: every bipartition has a part containing seven auxiliary points, which form a retained basis. That part spans, so the connectivity value is the rank of the other part, at least two whenever both parts have at least two elements.

We now count flags. Every selected block has the same old restriction

$$L = P_3(q) \oplus U_{4,4}.$$

Inside it, the old hyperplanes comprise $[3]_q = q^2 + q + 1$ copies of $P_2(q) \oplus U_{4,4}$ and four copies of $P_3(q) \oplus U_{3,3}$. Coalescence replaces these $[3]_q + 4$ hyperplanes by one and preserves all flats of rank at most 5. Thus the Whitney numbers are

$$W_s = \sum_{a+b=s} [6]_q \binom{N}{b} - \begin{cases} 0, & 1 \leq s \leq 5, \\ c([3]_q + 3), & s = 6. \end{cases}$$

Here $[d]_q$ counts a -dimensional subspaces of $\mathbb{F}_q^d$, and the sum ranges over $0 \leq a \leq 6$ and $0 \leq b \leq N$. For a rank set $S \subseteq \{1, \dots, d+n-1\}$, let $\alpha_{d,n}(S)$ count flags of proper flats in $P_d(q) \oplus U_{n,n}$, with $\alpha_{d,n}(\emptyset) = 1$. Counting by the largest flat gives the recurrence

$$\alpha_{d,n}(S) = \sum_{\substack{0 \leq a \leq d, 0 \leq b \leq n \\ a+b=s}} [d]_q \binom{n}{b} \alpha_{a,b}(S \setminus \{s\}), \quad s = \max S.$$

Proposition 4.16 For $T \subseteq \{1, \dots, 5\}$, the flag numbers of $\mathcal{C}(q, N)$ satisfy

$$\begin{aligned} \alpha_{\mathcal{C}(q,N)}(T) &= \alpha_{6,N}(T), \\ \alpha_{\mathcal{C}(q,N)}(T \cup \{6\}) &= \alpha_{6,N}(T \cup \{6\}) \\ &\quad + c(\alpha_{3,4}(T) - \alpha_{3,4}(T \cup \{6\})). \end{aligned}$$

Proof. Within each selected block, coalescence removes $\alpha_{3,4}(T \cup \{6\})$ flags ending at old hyperplanes and introduces $\alpha_{3,4}(T)$ flags ending at the new hyperplane. Distinct blocks contain no common old hyperplane, and these corrections are additive, so we obtain the desired count. $\square$

4.4. Multichain counts are not log-concave.

Proof of Theorem B. For each family $M = M(q)$ of rank r , define for $j = 1, 2, 3$,

$$c_j = \sum_{\substack{S \subseteq [r-1] \\ |S|=j}} \alpha_M(S), \quad D(q) = Z_M(3)^2 - Z_M(2)Z_M(4).$$

By (3), these counts yield multichain counts

$$\begin{aligned} Z_M(2) &= 2 + c_1, \\ Z_M(3) &= 3 + 3c_1 + c_2, \\ Z_M(4) &= 4 + 6c_1 + 4c_2 + c_3. \end{aligned} \tag{28}$$

Then the defect $D(q)$ can be viewed as a polynomial in q . We first determine its sign for $x \in \mathbb{R}_{>0}$. In each family, the leading coefficient of $D(q)$ is negative and all its other nonzero coefficients are positive. This implies that there is exactly one positive zero ξ , so $D(\xi) = 0$ and

$$D(x) > 0 \quad (0 < x < \xi), \quad D(x) < 0 \quad (x > \xi).$$

The matroids are defined only at the allowed prime powers q . Thus the threshold is the first allowed prime power greater than ξ . We locate this zero between consecutive allowed prime powers by exact evaluation, as follows.

- (a) For M_q , use Corollary 4.11 with $m = (q^2 + 1)/2$. The consecutive odd prime powers 89 and 97 satisfy

$$D(89) > 0 > D(97), \quad 89 < \xi < 97.$$

Hence, for odd prime powers q , $D(q) < 0$ if and only if $q \geq 97$.

- (b) For M_q^{pk} , use Corollary 4.11 with $m = q^2 + q + 1$. The consecutive prime powers 79 and $81 = 3^4$ satisfy

$$D(79) > 0 > D(81), \quad 79 < \xi < 81.$$

Hence, for prime powers q , $D(q) < 0$ if and only if $q \geq 81$.

- (c) For $G_{4,q}$, use the counts following Proposition 4.13 with $n = 6$, $m = q^3$, and $b = q^9$. Again, the consecutive prime powers 79 and 81 satisfy

$$D(79) > 0 > D(81), \quad 79 < \xi < 81.$$

Hence, for prime powers q , $D(q) < 0$ if and only if $q \geq 81$.

- (d) For $C(q, q^3)$, count chains by their largest proper flat. The ambient truncation has $\begin{bmatrix} 6 \\ a \end{bmatrix}_q \binom{q^3}{b}$ flats of type (a, b) , with $1 \leq a+b \leq 6$, each restricting to $P_a(q) \oplus U_{b,b}$. Repeat this count inside each restriction, then apply Lemma 3.9 to the $q^6 \binom{q^3}{4}$ blocks. The consecutive prime powers 139 and 149 satisfy

$$D(139) > 0 > D(149), \quad 139 < \xi < 149.$$

Hence, for prime powers q , $D(q) < 0$ if and only if $q \geq 149$.

Since $D(q) = Z_M(3)^2 - Z_M(2)Z_M(4)$, these are precisely the four equivalences in the theorem. Its final assertion follows from Theorem A. $\square$

4.5. Three-connected rank-7 counterexamples.

Proof of Theorem C. Let $X = M_q$ for an odd prime power q , or $X = G_{4,q}$ for any prime power q , and put

$$Y = \text{Tr}^2(X \oplus X \oplus U_{1,1}).$$

The direct sum is simple of rank 9 and hence its truncation is connected and simple, and its rank-7 truncation is simple by Lemma 3.2.

We count flags in $\mathcal{L}(X)^2 \times \mathcal{L}(U_{1,1})$ by their coordinate ranks, retaining total rank sets in [6]. For each choice of weakly increasing coordinate ranks, the chains in the factors are chosen independently. Substitution in Eqs. (18) and (19) gives the Chow coefficients $\underline{H}_i(q)$ and the cubic γ -polynomial Γ_q . Let

$$R(q) = \text{disc}(\Gamma_q), \quad \Delta(q) = \underline{H}_2(q)^2 - \underline{H}_1(q)\underline{H}_3(q).$$

The Taylor coefficients of $\gamma_3(q)$ at $q_{\min} = 3$ for the affine family, and at $q_{\min} = 2$ for the Gabidulin family, are nonnegative, with positive constant term. Hence $\gamma_3(q) > 0$ for $q \geq q_{\min}$, and Lemma 2.15 implies that

$$\begin{aligned} \underline{H}(Y; t) \text{ is not real-rooted} &\iff R(q) < 0, \\ \underline{H}(Y; t) \text{ is not log-concave} &\iff \Delta(q) < 0. \end{aligned}$$

We first determine the signs of R and Δ for real arguments $x \geq q_{\min}$. For either $D = R$ or $D = \Delta$, the polynomial $D(q_{\min} + x)$ has one sign variation, a positive constant term, and a negative leading coefficient. Descartes' rule therefore implies that $D(q)$ has exactly one zero $\xi_D > q_{\min}$, with

$$D(\xi_D) = 0, \quad D(x) > 0 \quad (q_{\min} \leq x < \xi_D), \quad D(x) < 0 \quad (x > \xi_D).$$

Thus ξ_R and ξ_Δ indicate where the discriminant and the log-concavity defect, respectively, change sign as q varies. We locate these zeros between consecutive allowed prime powers by computer computations.

- (a) For $X = M_q$, use Corollary 4.11 with $m = (q^2 + 1)/2$. The consecutive odd prime powers 43, 47 and 89, 97 satisfy

$$\begin{aligned} R(43) > 0 > R(47), \quad 43 < \xi_R < 47, \\ \Delta(89) > 0 > \Delta(97), \quad 89 < \xi_\Delta < 97. \end{aligned}$$

Hence, for odd prime powers q , $R(q) < 0$ if and only if $q \geq 47$, and $\Delta(q) < 0$ if and only if $q \geq 97$.

- (b) For $X = G_{4,q}$, use the counts following Proposition 4.13 with $n = 6$, $m = q^3$, and $b = q^9$. The consecutive prime powers $32 = 2^5$, 37 and $79, 81 = 3^4$ satisfy

$$\begin{aligned} R(32) > 0 > R(37), \quad 32 < \xi_R < 37, \\ \Delta(79) > 0 > \Delta(81), \quad 79 < \xi_\Delta < 81. \end{aligned}$$

Hence, for prime powers q , $R(q) < 0$ if and only if $q \geq 37$, and $\Delta(q) < 0$ if and only if $q \geq 81$.

By the two criteria above, these are the desired real-rootedness and log-concavity thresholds. $\square$

5. h -POLYNOMIALS OF GRAPH ASSOCIAHEDRA ARE NOT LOG-CONCAVE

In this section we construct a family of rooted graphs whose multichain counts fail log-concavity. These graphs consist of a long path, connected to a long periodic ladder via some sparse bridges. See Figure 5 for a schematic picture of these graphs. The path and the ladder offer two competing ways to go across the graph, which ultimately leads to a failure of log-concavity of the rooted multichain counts.

We first define the graphs and explain how their chain counts are computed exactly. We then give the counting heuristic which motivated the design. The exact counterexample in Section 5.4 is independent of this heuristic.

The failure of log-concavity is transferred to the h -polynomial via Theorem 2.14.

5.1. The graphs.

Definition 5.1 (Path and ladder) Fix a word $w = w_0 \cdots w_{L-1}$ in the letters I, O, N, a bridge pattern $\beta = (\beta_0, \dots, \beta_{L-1}) \in \{-1, 0, 1, 2, 3\}^L$, and an integer $R \geq 1$. The graph contains LR layers, indexed by $0 \leq i < LR$, with $b_i = \beta_{i \bmod L}$.

- (1) Each layer contains vertices a_i, c_i, c'_i and the edge $c_i c'_i$. If $b_i = -1$, there are no further edges within this layer. If $b_i = 0$, add $a_i c_i$ and $a_i c'_i$. If $b_i \geq 1$, add b_i new vertices to the layer, denoted $s_{i,1}, \dots, s_{i,b_i}$, together with the path $a_i s_{i,1} \cdots s_{i,b_i}$, and join s_{i,b_i} to both c_i and c'_i . We call this a bridge with b_i additional vertices.
- (2) Consecutive layers $i-1, i$ are joined by $a_{i-1} a_i$ and by ladder edges prescribed by $w_{(i-1) \bmod L}$:

$$\begin{array}{c|l} \text{I} & c_{i-1} c_i, c'_{i-1} c'_i \\ \text{O} & c'_{i-1} c_i, c'_{i-1} c'_i \\ \text{N} & \text{all four edges between } \{c_{i-1}, c'_{i-1}\} \text{ and } \{c_i, c'_i\}. \end{array}$$

See Figure 4.

- (3) A root p is joined to a_0, c_0, c'_0 . Optionally, choose $\ell \geq 0$, join a new vertex r to $a_{LR-1}, c_{LR-1}, c'_{LR-1}$, and attach a path $ru_1 \cdots u_\ell$. The resulting rooted graph is denoted $X(w, \beta; R, \ell)$. Without this optional attachment, r is absent. The resulting rooted graph is denoted $X(w, \beta; R, \ell)$.

A period contains

$$n_{\text{per}} = 3L + \sum_{r=0}^{L-1} \max(\beta_r, 0)$$

vertices. Thus the graph has $Rn_{\text{per}} + 1$ vertices without the attachment, and $Rn_{\text{per}} + 2 + \ell$ with it. All these graphs are connected, since each of the two tracks is connected to p .

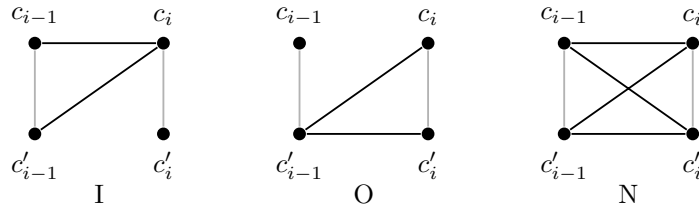


FIGURE 4. The three ladder interfaces. Grey edges are the fixed rungs within each layer.

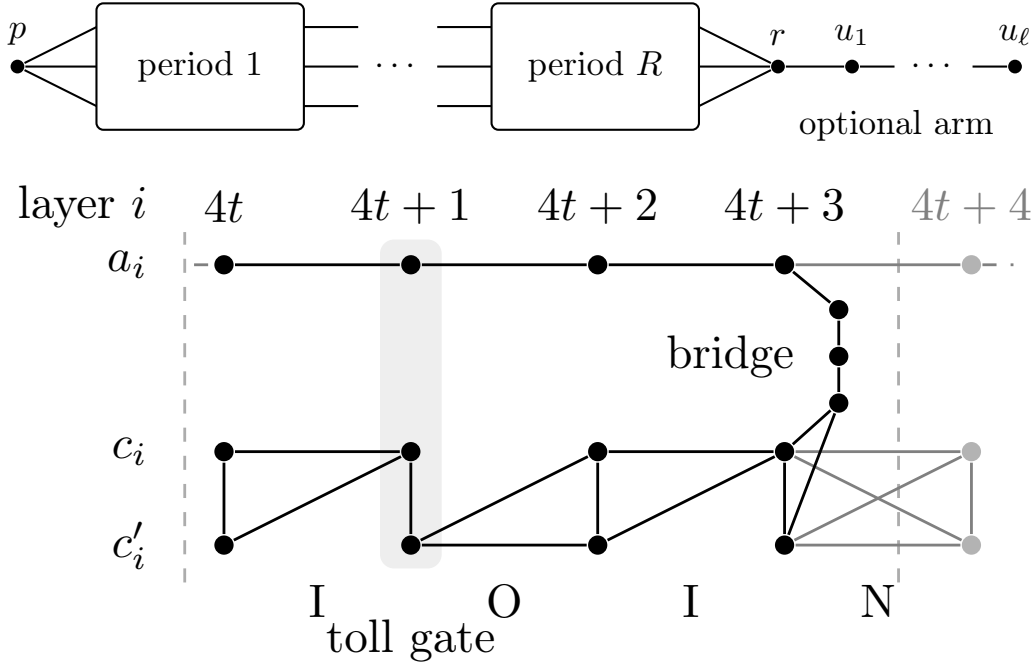


FIGURE 5. An example of a path and ladder. Above: the full strip with root p and optional terminal vertex r and arm; each box hides one period's internal edges. Below: the black vertices form one period; the grey layer starts the next. Dashed vertical lines mark period boundaries. The shaded layer is a toll gate: the incoming I edges reach only c_{4t+1} , whereas the outgoing O edges leave only from c'_{4t+1} . The bridge has three additional vertices.

The design which we will use for our explicit counter-example is IOIN/ $(-1, -1, -1, 3)$ (Figure 5). It has four layers and 15 vertices per period. At a layer entered through I and left through O, a crossing confined to the ladder must use both vertices. We call such a layer a *toll gate*. There is one toll gate per period in this design. The heuristics given in Section 5.3 explain why toll gates tend to produce failure of log-concavity of rooted multichain counts.

5.2. Computing multichain counts by a transfer matrix. Fix a design of Path and ladder graph, and a choice of terminal attachment, and write X_R for the resulting graph with R periods. We now explain how to compute $Z_{X_R}(j)$ exactly, providing the counts used in Section 5.4. The method processes one layer at a time and retains only the information needed to check connectivity. We call the long path and ladder, together with the bridges, the *strip*; the root and optional terminal attachment are handled at its two ends.

For $j \geq 2$, encode a multichain $\{p\} = S_0 \subseteq S_1 \subseteq \dots \subseteq S_j = V(X_R)$ by its *entry labels*

$$h(v) = \min\{t \in \{1, \dots, j\} : v \in S_t\}.$$

Thus $h(p) = 1$ by convention, and $S_t = \{v : h(v) \leq t\}$ for $1 \leq t \leq j$. Counting multichains amounts to counting labelings $h : V(X_R) \rightarrow \{1, \dots, j\}$ with $h(p) = 1$ for

which these sets are connected. Since the endpoints are prescribed and connected, only the thresholds $1 \leq t \leq j-1$ need to be checked.

We construct all possible entry labelings h layer by layer. After processing layer i , only a_i, c_i, c'_i can have neighbours in the unprocessed part. These three vertices form the *frontier*. A *state* is a record consisting of their labels and, for each threshold t , a partition of

$$\{p\} \cup \{v \text{ in the frontier} : h(v) \leq t\}.$$

Two vertices lie in the same block exactly when they are connected through processed vertices with labels at most t . This includes paths through vertices no longer in the frontier. We keep p as a permanent marker; its block records the component containing the root.

A *transfer* updates the counts of partial labelings in each state when a layer is added. Choose labels for the new vertices and, at each threshold t , merge blocks along new edges whose endpoints both have labels at most t . Then *forget* vertices outside the new frontier: remove them from the record, retaining the root marker. These vertices have no unprocessed neighbours. Reject an extension whenever a component not containing p loses its last frontier vertex: it can never connect to the root later, and so the labeling h will not correspond to a chain of connected subsets. The root component may lose all its frontier vertices, since S_t can be complete before the strip ends. Apply these rules at every threshold using the same vertex labels, and sum the counts of extensions giving the same new state. Bridge vertices are introduced and forgotten within their layer.

After processing the terminal attachment, if present, forget the last frontier, applying the same rejection rule. The accepted labelings are exactly the desired multichains: disconnected pieces are allowed temporarily, but each must eventually join the root component. For fixed j , there are only finitely many states, independently of R .

Proposition 5.2 *Fix a design (w, β) and a choice of terminal attachment, including its arm length when present. For each $j \geq 1$ there are a finite nonnegative integer matrix T_j and nonnegative integer vectors u_j, w_j , independent of R , such that*

$$Z_{X_R}(j) = u_j^\top T_j^R w_j \quad (R \geq 1). \quad (29)$$

Proof. For $j = 1$, there is only the chain $\{p\} \subseteq V(X_R)$, so $Z_{X_R}(1) = 1$; take $T_1 = (1)$ and $u_1 = w_1 = (1)$. For $j \geq 2$, group L successive layer transfers into one period. At every period boundary, the frontier has the same three positions and the next period has the same edge pattern. Let $A_j(\sigma, \tau)$ count the allowed labelings of one further period taking state σ to state τ . Let $v_j(\sigma)$ count initializations through the first period, including the root edges, and let $\omega_j(\sigma)$ count accepted terminal extensions, including r and its arm when present. These are nonnegative integers, and summing over successive states gives $Z_{X_R}(j) = v_j^\top A_j^{R-1} \omega_j$.

To obtain the exponent R in (29), add one start state $*$, put $T_j(*, \sigma) = v_j(\sigma)$ and $T_j(\sigma, \tau) = A_j(\sigma, \tau)$, and allow no transition into $*$. Take u_j supported at $*$ with value 1, and set $w_j(*) = 0$ and $w_j(\sigma) = \omega_j(\sigma)$. Then $u_j^\top T_j^R w_j = v_j^\top A_j^{R-1} \omega_j$, as required. $\square$

5.3. Why the two parts compete. This subsection is heuristic. Fix $j \geq 2$ and consider chains whose innermost intermediate set S_1 crosses many periods. A small crossing along the path uses one vertex in each layer. A ladder crossing has more

route choices, but must pay for the toll gate with one extra vertex per period. For the design in Figure 5, the two crossings therefore use respectively 4 and 5 of the 15 vertices in a period.

Once S_1 is fixed, its vertices have label 1. Each vertex outside S_1 has $j - 1$ possible entry labels, namely $2, \dots, j$. Consequently, a minimal path crossing leaves 11 labels free, whereas a minimal ladder crossing leaves 10. This suggests per-period contributions of the forms

$$m_P(j - 1)^{11} \quad \text{and} \quad m_L(j - 1)^{10}. \quad (30)$$

The coefficients absorb route choices and connectivity constraints; the free labels cannot in general be chosen independently.

For example, suppose s new vertices form a path attached to a vertex in S_1 , with all s vertices outside S_1 and no other connection to the graph. Its labels, ordered away from the attachment, must satisfy

$$2 \leq h_1 \leq \dots \leq h_s \leq j.$$

There are $\binom{j+s-2}{s} \sim (j-1)^s/s!$ choices. Thus the exponent still counts free vertices, but the leading coefficient can be fractional. The quantities m_P, m_L in (30) should be understood in this sense, rather than as integer numbers of routes. In the ladder regime, multiple routes help compensate for the lost power of $j - 1$.

More generally, if a path crossing leaves e free vertices and the ladder pays a toll of Δ extra vertices per period, the comparison is between $m_P(j - 1)^e$ and $m_L(j - 1)^{e-\Delta}$. When $m_L > m_P$, their balance occurs at

$$j_* = 1 + (m_L/m_P)^{1/\Delta}.$$

Below this scale the ladder's route choices can dominate; above it, the path's extra free labels can dominate. Here $\Delta = 1$.

This change can favour log-convexity even though each separate power of $j - 1$ is log-concave. Indeed, the logarithmic slopes of the two expressions are respectively $(e - \Delta)/(j - 1)$ and $e/(j - 1)$. The dominant contribution therefore changes towards a larger slope at the balance point, which can lead to a failure of log-concavity for multichain counts. Repeating the period many times reinforces the competition.

Finally, sparse bridges make it costly for a crossing to change tracks. To pass between the path and ladder within a period, S_1 must contain an entire bridge. Each additional bridge vertex then has label 1 instead of approximately $j - 1$ later-entry choices. Long enough bridges, placed only in a few layers, can therefore suppress mixed crossings and preserve the competition between the two regimes. Whether the resulting finite graph actually fails log-concavity is decided by the exact transfer computation, as in the certificate below.

5.4. A counterexample. We now fix one graph and leave the heuristic behind. Take

$$G = X(\text{IOIN}, (-1, -1, -1, 3); 98, 68),$$

rooted at p , with the terminal vertex and arm included. There are 392 layers, 98 bridges with three new vertices each, and 68 arm vertices. Consequently

$$|V(G)| = 2 + 3 \cdot 392 + 3 \cdot 98 + 68 = 1540.$$

The graph has 2323 edges. Its toll layers are precisely those with index $i \equiv 1 \pmod{4}$; its bridges lie in layers $i \equiv 3 \pmod{4}$.

Theorem 5.3 *For this rooted graph G ,*

$$Z_G(37)^2 < Z_G(36)Z_G(38).$$

For every integer $q \geq 23$, the graph $W_q(G)$ satisfies

$$h_{36}(W_q(G))^2 < h_{35}(W_q(G))h_{37}(W_q(G)).$$

Thus its h -polynomial is neither log-concave nor real-rooted. The graph $W_{23}(G)$ has $1 + 23 \cdot 1539 = 35,398$ vertices.

5.4.1. *Computational input and its verification.* The input consists of the integer chain counts $Z_G(j)$ for $1 \leq j \leq 38$ and the integer $t_G = |\mathcal{B}(G - p)|$. The certificate below uses only coarse inequalities implied by these computations. Put

$$\rho = \frac{Z_G(37)^2}{Z_G(36)Z_G(38)}.$$

The required inputs are

$$\begin{aligned} t_G &< 2^{881}, & Z_G(a+1) &> 10^{12} Z_G(a) \quad (1 \leq a < 38), \\ \rho &< \frac{9998}{10000}. \end{aligned} \tag{31}$$

The three central integers have 1347, 1359, and 1371 decimal digits, respectively. To indicate their scale, the calculation gives

$$\begin{aligned} \log_{10} Z_G(36) &\approx 1346.6221, \\ \log_{10} Z_G(37) &\approx 1358.6728, & \log \rho &\approx -0.0002694. \\ \log_{10} Z_G(38) &\approx 1370.7236, \end{aligned}$$

These decimal displays are illustrations, not substitutes for the integer comparisons in (31).

The chain counts were obtained with multiprecision integer arithmetic by a transfer over the layers. An independently written program used directed rounding to give upper bounds for all $1 \leq j \leq 38$ and lower bounds for $j = 36, 37, 38$. The resulting brackets, of relative width at most $5.9 \cdot 10^{-12}$, contained the exact integers. The number t_G was counted by two independent frontier programs; it has 881 bits. The implementations were also checked against direct enumeration and generic frontier computations on smaller test instances.

For the layer transfer, bridge vertices are summed out inside their layer. The frontier entry times and connectivity data are those described in Section 5.2; the terminal attachment and arm are included in the final weights. All operations in the exact count are sums and products of nonnegative integers. In the interval implementation, the corresponding operations are rounded downwards and upwards to enclose the count. No asymptotic estimate of a channel growth rate is used in this computation.

5.4.2. *A short certificate for every $q \geq 23$.*

Proof of Theorem 5.3 from the computational input. The last inequality of (31) already proves the zeta failure. It remains to apply the error bound of Theorem 2.13. Set $n = 1540$ and $c = n - 1 + t_G$. By (31), $c < 2^{882}$. For $j \in \{35, 36, 37\}$ and $m = j - a \geq 1$, successive multiplication of the zeta-ratio bounds gives

$$\frac{Z_G(j+1-m)}{Z_G(j+1)} < 10^{-12m}.$$

Since $23 \cdot 2^{882} + 37 < 10^{268}$, the error at $q = 23$ satisfies

$$\begin{aligned} E_j(23) &= \sum_{m=1}^j \left(\frac{Z_G(j+1-m)}{Z_G(j+1)} \right)^{23} \binom{23c+m}{m} \\ &< \sum_{m=1}^j 10^{-276m} (23c+m)^m < \sum_{m \geq 1} 10^{-8m} < 2 \cdot 10^{-8}. \end{aligned}$$

Writing $\varepsilon = 2 \cdot 10^{-8}$, exact rational arithmetic gives

$$\rho^{23} \frac{(1 + E_{36}(23))^2}{(1 - E_{35}(23))(1 - E_{37}(23))} < \left(\frac{9998}{10000} \right)^{23} \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2} < 1.$$

This proves the h -inequality for $q = 23$.

To cover all larger integers, consider an individual error summand $r^q \binom{qc+m}{m}$, where $r = Z_G(j+1-m)/Z_G(j+1) < 10^{-12m}$. Its ratio at $q+1$ and q is

$$r \prod_{s=1}^m \frac{(q+1)c+s}{qc+s} \leq r \left(1 + \frac{1}{q} \right)^m < 10^{-12m} \left(\frac{24}{23} \right)^m < 1 \quad (q \geq 23).$$

Thus every $E_j(q)$ decreases, as does ρ^q . The amplification criterion remains valid for every $q \geq 23$. Newton's inequalities exclude real-rootedness, and the vertex count follows from the definition of a wedge. $\square$

The value 23 is a certified sufficient number of copies. This argument does not determine the first q for which the family fails log-concavity. In particular, a zeta failure for the single rooted core is not by itself a certificate of an h -failure for that core.

Table 1 below gives an account of other patterns leading to failures of log-concavity of multichain counts. Each entry describes the ladder pattern, the number of periods used, the size of the arm, the size of the path and ladder graph, the index of the failure, the natural logarithm of $\rho = Z_X(k)^2/(Z_X(k-1)Z_X(k+1))$, the big length of t_X , the number of wedges used to produce a failure for h . The status entry specifies if the computations have been verified once or twice.

TABLE 1. Failures of log-concavity for other patterns

design w/β	R	arm	$ X $	k	$\log \rho$	t_X	bits	q_0	$ W_{q_0}(X) $	status
IOIN/ $(-1, -1, -1, 3)$	98	68	1540	36	-0.000269	881	23	35,398	twice	
IOIN/ $(-1, -1, -1, 3)$	98	75	1547	36	-0.002527	881	23	35,559	twice	
IONI/ $(-1, -1, -1, 3)$	86	157	1449	43	-0.000189	776	26	37,649	once	
IONI/ $(-1, -1, -1, 3)$	87	150	1457	43	-0.002464	785	26	37,857	once	
IONI/ $(-1, -1, -1, 3)$	88	150	1472	43	-0.006927	794	26	38,247	twice	
IONI/ $(-1, -1, -1, 3)$	86	200	1492	43	-0.007713	776	26	38,767	twice	
IONN/ $(-1, -1, -1, 2)$	189	200	2848	29	-0.002362	1731	21	59,788	once	
IONN/ $(-1, -1, -1, 2)$	190	200	2862	29	-0.004740	1740	21	60,082	twice	
IONN/ $(3, -1, 3, -1)$	130	200	2542	44	-0.003952	1633	29	73,690	twice	
IONN/ $(3, -1, 3, -1)$	188	none	3385	45	-0.000574	2349	31	104,905	twice	

REFERENCES

- [AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452, arXiv:1511.02888. [2](#)
- [AKE23] Christos A. Athanasiadis and Katerina Kalampoglia-Evangelinou, *Chain enumeration, partition lattices and polynomials with only real roots*, Comb. Theory **3** (2023), no. 1, Paper No. 12, arXiv:2205.03796. [3](#)
- [BHM⁺22] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, *A semi-small decomposition of the Chow ring of a matroid*, Adv. Math. **409** (2022), Paper No. 108646, arXiv:2002.03341. [1](#), [2](#)
- [Bjö92] Anders Björner, *The homology and shellability of matroids and geometric lattices*, Matroid Applications, Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, 1992, pp. 226–283. [9](#)
- [BL25] Petter Brändén and Leonardo Saud Maia Leite, *On chain polynomials of geometric lattices*, 2025, arXiv:2508.13810v3. [3](#), [6](#)
- [Brä06] Petter Brändén, *On linear transformations preserving the Pólya frequency property*, Transactions of the American Mathematical Society **358** (2006), no. 8, 3697–3716. [7](#)
- [Brä15] ———, *Unimodality, log-concavity, real-rootedness and beyond*, Handbook of Enumerative Combinatorics (Miklós Bóna, ed.), Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015, pp. 437–483. [2](#), [14](#), [19](#)
- [BV25a] Petter Brändén and Lorenzo Vecchi, *Chow polynomials of totally nonnegative matrices and posets*, 2025, arXiv:2509.17852v3. [3](#)
- [BV25b] ———, *Chow polynomials of uniform matroids are real-rooted*, 2025, arXiv:2501.07364v2. [3](#)
- [BW96] Anders Björner and Michelle Wachs, *Shellable nonpure complexes and posets. i*, Transactions of the American mathematical society **348** (1996), no. 4, 1299–1327. [2](#)
- [CD04] Michael Carr and Satyan L. Devadoss, *Coxeter complexes and graph-associahedra*, Topology and its Applications, 153 (2006) 2155–2168 (2004). [7](#)
- [CFL25] Basile Coron, Luis Ferroni, and Shiyue Li, *Chow polynomials of rank-uniform labeled posets*, 2025, arXiv:2511.13819v2. [3](#)
- [CFL26a] ———, *Matroid analogues of Gal’s conjecture*, arXiv e-prints, 2026, arXiv:2604.04550v3. [3](#), [6](#)
- [CFL26b] ———, *Structural properties of nested set complexes*, 2026, arXiv:2601.05188v2. [2](#)
- [Cra70] Henry H. Crapo, *Erecting geometries*, Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications (Chapel Hill), University of North Carolina, 1970, pp. 74–99. [4](#), [20](#), [21](#)
- [Del78] Ph. Delsarte, *Bilinear forms over a finite field, with applications to coding theory*, J. Combin. Theory Ser. A **25** (1978), no. 3, 226–241. [4](#)
- [Dem68] Peter Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 44, Springer-Verlag, Berlin, 1968. [26](#)
- [Den72] R. H. F. Denniston, *Some packings of projective spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **52** (1972), 36–40. [4](#), [30](#)
- [Den73] ———, *Packings of $PG(3, q)$* , Finite Geometric Structures and their Applications (Bresanone, 1972), Edizioni Cremonese, 1973, pp. 193–199. [4](#), [30](#)
- [Etz15] Tuvi Etzion, *Partial k -parallelisms in finite projective spaces*, Journal of Combinatorial Designs **23** (2015), no. 3, 101–114. [26](#)
- [Fer26] Luis Ferroni, *Unimodality shenanigans in ehrhart theory*, arXiv preprint arXiv:2609.10513 (2026). [3](#)
- [FH24] Luis Ferroni and Akihiro Higashitani, *Examples and counterexamples in Ehrhart theory*, EMS Surveys in Mathematical Sciences (2024), Published online first. [3](#)
- [FMSV24] Luis Ferroni, Jacob P. Matherne, Matthew Stevens, and Lorenzo Vecchi, *Hilbert–Poincaré series of matroid Chow rings and intersection cohomology*, Adv. Math. **449** (2024), Paper No. 109733, arXiv:2212.03190. [2](#), [3](#), [10](#)
- [FMV24] Luis Ferroni, Jacob P. Matherne, and Lorenzo Vecchi, *Chow functions for partially ordered sets*, arXiv e-prints, 2024, arXiv:2411.04070v3. [11](#), [12](#), [19](#)

- [FNV22] Luis Ferroni, George D. Nasr, and Lorenzo Vecchi, *Stressed hyperplanes and Kazhdan–Lusztig gamma-positivity for matroids*, Int. Math. Res. Not. IMRN (2022), rnac270, doi:10.1093/imrn/rnac270, arXiv:2110.08869. 21
- [Fro10] Georg Frobenius, *Über die Bernoullischen Zahlen und die Eulerschen Polynome*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1910), 809–847. 7
- [FS24] Luis Ferroni and Benjamin Schröter, *Valuative invariants for large classes of matroids*, J. Lond. Math. Soc. (2) **110** (2024), no. 3, e12984, arXiv:2208.04893. 2
- [FY04] Eva Maria Feichtner and Sergey Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*, Invent. Math. **155** (2004), no. 3, 515–536. MR 2038195 1
- [Gab85] E. M. Gabidulin, *Theory of codes with maximum rank distance*, Problems Inform. Transmission **21** (1985), no. 1, 1–12, Translated from Problemy Peredachi Informatsii **21** (1985), no. 1, 3–16. 4
- [Hir85] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1985. 4, 27, 28
- [Hir98] ———, *Projective geometries over finite fields*, 2 ed., Oxford Mathematical Monographs, Oxford University Press, Oxford, 1998. 26
- [HS25] Elena Hoster and Christian Stump, *Chow polynomials of simplicial posets with positive h -vector are real-rooted*, 2025, arXiv:2508.15538v1. 3
- [HZ19] James Haglund and Philip B. Zhang, *Real-rootedness of variations of Eulerian polynomials*, Advances in Applied Mathematics **109** (2019), 38–54. 7
- [Kle84] Felix Klein, *Ueber die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Coordinationen auf eine canonische Form*, Math. Ann. **23** (1884), 539–578, Revised reprint of the author’s 1868 Bonn dissertation. 27
- [Knu75] Donald E. Knuth, *Random matroids*, Discrete Math. **12** (1975), no. 4, 341–358. 4, 21
- [New07] Isaac Newton, *Arithmetica universalis: sive de compositione et resolutione arithmetica liber*, Typis Academicis, Cambridge; Benjamin Tooke, London, 1707, Edited by William Whiston. 2, 14, 19
- [Oxl11] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, 2011. 20, 21
- [Pet15] T Kyle Petersen, *Eulerian numbers*, Eulerian Numbers, Springer, 2015, pp. 3–18. 20
- [Pie25] Paweł Pielasa, *Decompositions of Chow rings of direct sums of matroids*, 2025, arXiv:2511.10746. 14
- [PRW08] Alexander Postnikov, Victor Reiner, and Lauren Williams, *Faces of generalized permutohedra*, Doc. Math. **13** (2008), 207–273. 7, 15
- [PvdP17] Rudi Pendavingh and Jorn van der Pol, *Enumerating matroids of fixed rank*, Electron. J. Combin. **24** (2017), no. 1, Paper No. 1.8, arXiv:1512.06655. 4, 20, 21, 22
- [Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. 3, 9, 11
- [Ste21] Matthew Stevens, *Real-rootedness conjectures in matroid theory*, Bachelor’s thesis, Stanford University, 2021, <https://mcs0042.github.io/bachelor.pdf>. 2
- [Stu24] Christian Stump, *Chow and augmented Chow polynomials as evaluations of Poincaré-extended ab -indices*, 2024, arXiv:2406.18932v3, to appear in Adv. Math. 11, 19
- [Zel06] Andrei Zelevinsky, *Nested complexes and their polyhedral realizations*, Pure and Applied Mathematics Quarterly **2** (2006), no. 3, 655–671. 15

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