Lecture Notes on Trees

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Contents

In this class, we will use **trees** to answer the following questions:

- How can you have a seemingly "bigger group" as a subgroup of a seemingly "smaller group"?
- Can you decompose a ball into a finite number of sets and reassemble them into two balls identical to the original?
- How do you describe behaviors of various kinds infinite groups?

These questions and solutions to these questions fall into the realm of geometric group theory where mathematicians have employed a beautiful mixture of algebraic and geometric methods to understand not only the underlying geometry but also the groups associated with these spaces. The spaces that we will focus on in this class are **trees** that possess symmetries of groups. We will see how studying these particular trees can reflect properties of various groups and yield many fruitful results.

1 Growing Trees

Goal 1.1. Our goal for Day 1 is to understand the following:

- the construction of Cayley graphs of groups and examples of them;
- the definition and concepts of free groups;
- the intuition and the reason why one can have a seemingly "bigger group" as a subgroup of a smaller group; in particular, we will present a proof of "any free group of rank 2 can have a free group of rank 3 as a subgroup".

1.1 Graphs and Trees

Definition 1.2. A **graph** G is an pair (V, E) where V is a set of vertices, and E is a set of edges, or 2-element subsets of V. If each edge is an ordered pair, we say that G is a **directed graph**; if not, we say that G is an **undirected graph**.

Definition 1.3. A **tree** is an undirected graph where every pair of vertices has exactly one path connecting them. In other words, a tree is an **acyclic connected graph**.

Question 1.4. Given a group, how do we construct a graph that reflects the algebraic relations in the group?

1.2 Cayley Graphs

Definition 1.5. A **generating set** of a group is a subset such that every element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset and their inverses. A group G is **finitely generated** if it has a finite generating set.

Definition 1.6. G is a group and S be a generating set. The Cayley graph $\Gamma = \Gamma(G, S)$ is a colored directed graph that has the following property:

- Every element g of G is represented by a vertex in Γ.
- Every generator $s \in S$ is assigned a color.
- For any $g \in G, s \in S$, there is an edge (g, gs) colored with the color of s.

Question 1.7. Is Cayley graph associated with any group unique?

Example 1.8.

- What is a Cayley graph of C_2, the cyclic group of order 2?
- What is a Cayley graph of C_3, the cyclic group of order 3?
- What is a Cayley graph of the additive group $\mathbb{Z}$?
- What is the Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$?

Question 1.9. A **cycle** is a path of edges and vertices wherein a vertex is reachable from itself. Are there cycles in the Cayley graphs we constructed above? What causes the cycles?

1.3 Removing Cycles

Game 1.10. You and your friend is trying to construct words in this language using l, o and their inverses l^{-1}, o^{-1}. To construct a word, you each take turns to put down l or o. To prevent the words from getting too long, we have the reduction rule that

$$ll^{-1} = l^{-1}l = \varnothing, oo^{-1} = o^{-1}o = \varnothing.$$

Once you finish constructing a word and reducing it to the most simplified form, you are done.

Example 1.11. We denote the set of reduced words generated by $\{l, o\}$ as $W(\{l, o\})$ or W. Then $lol \in W$, so is $lolololol$. Note that $lol = lo^{-1}ool = lo^{-1}oolo^{-1}o \in W$.

Question 1.12. Is $W(l, o)$ a group? If so, can you draw its Cayley graph?

Definition 1.13. The set of all reduced words W constructed from $S = \{l, o\}$ and their inverses is called a **free group** with basis S. We say that S is the generating set of W.

Example 1.14. Are the following groups free groups? If so, with what basis?

- The cyclic group C_3.
- The additive group $\mathbb{Z}$.
- The additive group $4\mathbb{Z} \oplus 3\mathbb{Z}$.

Definition 1.15. The **rank** of a free group with basis S is the number of elements in S. We denote a free group of rank n by F_n.

Question 1.16. Given any free group F, what is the **order** of F, i.e. the number of elements in F?

1.4 F_3 as Subgroup of F_2

Now we start to contemplate on relations between free groups of different ranks.

Question 1.17. Does F_3 have a subgroup that is isomorphic to a F_2?

It is not surprising that F_2 is a subgroup of F_n for all $n \geq 2$. But the following result is rather bizzare.

Theorem 1.18. *There exists a finite index subgroup of F_2 that is a free group of rank 3.*

Definition 1.19. For every $g \in F_2$, the **length** of g is the number of terms in its reduced form, denoted by $|g|$.

Question 1.20. For any $g \in W$, what is the geometric meaning of $|g|$ on $\Gamma(W, \{l, o\})$ in Figure 1?

Definition 1.21. Let H be the subset of F_2 consisting of elements of even length:

$$H = \{g \in F_2 : |g| \text{ is even }\}.$$

We call H the **even subgroup** of F_2.

We start by showing the following lemmas.

Lemma 1.22. *The even subgroup H of W can be generated by $T = \{l^2, lo, lo^{-1}\}$.*

Proof. Exercise [1]. □

[1] Hint: Start by showing that every reduced word of length 2 can be expressed as a product of elements of $S \cup S^{-1}$

Lemma 1.23. *Any reduced word w generated by $T = \{l^2, lo, lo^{-1}\}$ is not the identity in W.*

Proof. Let $w = w_1 w_2 \dots w_2$ be a reduced word in W. Note that w written in original generators $T = \{l, o\}$ of W may cause cancellation to occur. For example, w might be $(l^{-2})(lo) = l^{-1}l^{-1}lo = l^{-1}o$.

Similarly, we might have

$$(lo^{-1})^{-1}(ll)(lo^{-1})^{-1}(lo) = \underline{ol^{-1}ll}\ \underline{ol^{-1}lo} = \underline{ol}\ \underline{oo}.$$

But notice that in this example, there is cancellation in $(lo^{-1})^{-1}(ll)$ and $(lo^{-1})^{-1}(lo)$, but there is no cancellation in $(ll)(lo^{-1})^{-1}$ in terms of original generators S.

We need to think more carefully about what happens when replacing each w_i in w with $x_i y_i$ where $x_i, y_i \in S$. In the example above,

$$w_1 w_2 w_3 w_4 = (x_1 y_1)(x_2 y_2)(x_3 y_3)(x_4 y_4) = (lo^{-1})^{-1}(ll)(lo^{-1})^{-1}(lo),$$

we see that $y_1 = l^{-1} = x_2$ and $y_3 = l^{-1}x_4$. We will see that these internal cancellations are the worst that can happen. That is, even if $y_i = x_{i+1}^{-1}$, then the other parts of the word w will not collapse that much. In particular, we will show that (Exercise): if $y_i = x_{i+1}^{-1}$, then the following things hold

- $x_i \neq y_{i+1}^{-1}$;
- $y_{i+1} \neq x_{i+2}^{-1}$;
- $y_{i-1} \neq x_i^{-1}$.

This implies that, given a reduced word $w = w_1 \cdots w_n = x_1 y_1 x_2 y_2 \cdots x_n y_n \in H$, when written in terms of generators S, at most one of of the letters x_i or y_i will be canceled from each $x_i y_i = w_i$. Thus, the length of $w \in W$ in terms of S,

$$|w| = |x_1 y_1 x_2 y_2 \cdots x_n y_n| \geq n \neq 0.$$

□

Complete the proof using the lemmas (Exercise). In fact, this is only the beginning of all the surprises you will encounter in this class.

Theorem 1.24. *The free group F_2 has a subgroup isomorphic to the free group F_m for all $m > 2$.*

Theorem 1.25. *The free group F_2 has a subgroup isomorphic to the free group F_∞ of countable rank.*

1.5 Exercises for Day 1

Exercise 1.26. Show that every finite group is finitely generated.

Exercise 1.27. This exercise gives you an example of non-uniqueness of Cayley graphs of a given group.

- Construt a Cayley graph of the dihedral group D_4, with respect to a generating set consisting of a rotation and a reflection.
- Construt a Cayley graph of the dihedral group D_4, with respect to a generating set consisting of two adjacent reflections.

Exercise 1.28. Is $\mathbb{Q}$, the rationals, a free group? If so, with what basis? If not, what is wrong?

Exercise 1.29. Show that even subgroup H of F_2 can be generated by $S = \{l^2, lo, lo^{-1}\}$.

Exercise 1.30. Show that: if $y_i = x_{i+1}^{-1}$, then the following things hold

- $x_i \neq y_{i+1}^{-1}$;
- $y_{i+1} \neq x_{i+2}^{-1}$;
- $y_{i-1} \neq x_i^{-1}$.

Exercise 1.31. Use the lemmae to show that $H \cong F_3$ is a subgroup of F_2.

2 Groups Action

Goal 2.1. Yesterday we saw how to use trees to gain intuition about free group structure, in particular, the reason why the free group of rank 2 can contain a subgroup of higher rank. We now move on to prove Banach-Tarski Theorem free group actions and trees. The goal of Day 2 is to understand the following things that will set the foundation for the arguments.

- a group acting on their Cayley graphs;
- Cayley's theorem;
- orbit and stablizer.

Definition 2.2. A symmetry of a graph Γ is a bijection σ taking vertices to vertices and edges to edges such that if $(u, v) \in E$, $\sigma((u, v)) = (\sigma(u), \sigma(v))$. The set of symmetries of a graph forms a group.

Example 2.3.

- What is the symmetry group of the complete graph K_n?
- What is the symmetry group of the following graph?
- What is the group of symmetry of the Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$?
- What is a group of symmetry of the Cayley graph of F_2?

Definition 2.4. An action of a group G on a geometric object X is a map ϕ from G to $\mathrm{Sym}(X)$ such that

$$g \cdot x = \phi(g)(x).$$

Example 2.5. In the Cayley graph of F_2, fix a vertex v_h, that is joined to v_{hl}. Then for any elementy $g \in F_2$, $v_{g \cdot h}$ is joined by l to $v_{g \cdot hl}$

Theorem 2.6. *Every group acts on its Cayley graph.*

Example 2.7. Consider the Cayley graph $\Gamma(D_4, \{r, s\})$ where r is the rotation and s is the reflection. How does D_4 act on its Cayley graph?

Definition 2.8. Let G acts on X. If $x \in X$, then the stabilizer of x, is

$$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}.$$

For any $x \in X$, $\mathrm{Stab}(x)$ is a subgroup of G.

Example 2.9. Let D_4 acts on a square. What is the stabilizer of a vertex?

Definition 2.10. Let G acts on X and let $x \in X$. Then the orbit of x is

$$\mathrm{Orb}\, x = \{g \cdot x : g \in G\}.$$

Example 2.11. What is the orbit of a vertex when D_4 acts on a square?

Definition 2.12. A group action is free if $g \cdot x = x$ implies that $g = e_G$. Equivalently, the stablizer for any x is trivial.

Proposition 2.13. *Any group acts freely on its Cayley graph.*

Theorem 2.14. *A group is free if and only if it acts freely on a tree.*

Theorem 2.15 (Nielsen-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. Let H be a subgroup of a free group F. F acts freely on any tree. Since H is a subgroup of F, H acts freely on any tree. Hence H is free. $\square$

2.1 Exercises for Day 2

Exercise 2.16.

- What is the symmetry group of the complete graph K_n?
- What is the symmetry group of a square with
- What is the symmetry group of the Cayley graph $\Gamma(\mathbb{Z}, \{(0,1),(1,0)\})$ of $\mathbb{Z} \oplus \mathbb{Z}$?
- What is the symmetry group of the Cayley graph $\Gamma(F_2, \{l, o\})$ of F_2?

Exercise 2.17. This exercise shows you a path that a vertex in the Cayley graph $\Gamma(F_2, \{l, o\})$ takes under an element of F_2 is not the path spelled out by the generaters. Trace the path of the action of $lol^{-1}o^{-1}$ on the vertex corresponding to the identity of F_2. Think carefully about where each edge and vertex goes. Pay attention to the fact that F_2 acts on its Cayley graph from the left.

Exercise 2.18. Let D_4 acts on a square. What is the stabilizer of a vertex? What is the orbit of a vertex?

Exercise 2.19. How does D_4 acts on its Cayley graph with respect to the generating set consisting of a rotation and a reflection?

Exercise 2.20. Show that every group acts freely on its Cayley graph.

Exercise 2.21 (Challenge)**.** What is the symmetry group of a cube? [2]

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[2] Hint: Think about the following fact: The group symmetry of a cube is the same as the automorphism group of the Euclidean space $\mathbb{R}^3$. What is the automorphism group of $\mathbb{R}^3$?

Recall the definition of Cayley graph.

Definition 2.22. G is a group and S be a generating set. The Cayley graph $\Gamma = \Gamma(G, S)$ is a colored directed graph that has the following property:

- Every element g of G is represented by a vertex in Γ.
- Every generator $s \in S$ is assigned a color.
- For any $g \in G$, $s \in S$, there is a directed edge (g, sg) colored with the color of s.

Definition 2.23. An action of a group G on a geometric object X is a map ϕ from G to $\mathrm{Sym}(X)$ such that

$$g \cdot x = \phi(g)(x).$$

Definition 2.24. Given a group G, and its Cayley graph $\Gamma = (V, E)$, G acts on the right of vertex by right multiplication. The action is defined as

$$x \cdot g = xg,$$

for any $x \in V$ and $g \in G$. This will satisfy the group action axiom

$$x \cdot e_G = x \text{ and } x \cdot (gh) = (xg) \cdot h = xgh$$

for all $g, h \in G$.

Example 2.25. For any generating element s of Γ, if v_x is connected with v_{sx}, then under the action of any gh, v_{xgh} is connected with v_{sxgh}.

Definition 2.26. A group action is free if for all x, $g \cdot x = x$ implies that $g = e_G$. Equivalently, the stablizer for any x is trivial.

Proposition 2.27. *Any group acts freely on its Cayley graph.*

Proof. Exercise. □

Theorem 2.28. *A group is free if and only if it acts freely on a tree.*

Proof. Exercise. □

Theorem 2.29 (Nielsen-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. Let H be a subgroup of a free group F. F acts freely on any tree. Since H is a subgroup of F, H acts freely on any tree. Hence H is free. □

2.2 How Else Can You Define A Group?

We have seen the Cayley graphs of "lol"-language, D_4, $\mathbb{Z} \oplus \mathbb{Z}$ with respect to some obvious generating sets. In some of them, you are able to reach one vertex via two distinct path; in some of them, you cannot. The reason behind is that, the distinct paths define relations of the generators in a group.

Definition 2.30. Given a group G, let S be the set of generators so that every element of the group can be written as a combination of generators under the group operation, and let R be the set of relations among those generators. We then say G has presentation

$$\langle S \mid R \rangle.$$

Example 2.31. How do you represent the following groups using generators and relations?

- The cyclic group of order 5, C_5;
- Dihedral group D_4;
- $\mathbb{Z} \oplus \mathbb{Z}$;
- Free group F_2.
- $\mathbb{Z}_n \times \mathbb{Z}_m$ for $m, n \in \mathbb{Z}$.

Definition 2.32. A **free group** generated of rank n is

$$\langle x_1, x_2, \ldots, x_n \rangle.$$

It is called "free" because there is no relations. Yay, freedom!

2.3 Exercises for Day 3

Shiyue consulted [MC17] when writing these problems.

Exercise 2.33. How do you represent the group $\mathbb{Z}_n \times \mathbb{Z}_m$ for natural numbers n, m as generators and relations?

These exercises let you play with the ideas of matrix groups, which will be useful in proving Banach-Tarski Theorem.

Exercise 2.34. Show that the group $G = \mathrm{SL}_2(\mathbb{Z})$ (the group of 2×2 matrices with integer entries and determinant 1) is not a free group. [3]

Exercise 2.35. How do you define $\mathrm{SL}_2(\mathbb{Z})$ using generators and relations?

The following exercise helps you work with group action in $\mathbb{R}^3$ and using geometric arguments to understand groups. Before we do so, let us see a new definition.

Definition 2.36. An map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry if for any $x, y \in \mathbb{R}^3$, the distance between x, y is the same as distance between $f(x)$ and $f(y)$.

Exercise 2.37. We want to show the following theorem step by step. Suppose that G is a group that has a free action by isometry on $\mathbb{R}^3$. Then G does not have elements of finite order.

(a) Write out what it means to say that G acts by isometry on $\mathbb{R}^3$.

(b) Assume that G has a element of finite order, g. Consider the subgroup generated by g. Fix an $x \in \mathbb{R}^3$. What is the orbit O_x of some x under the action of g.

(c) Use the fact that every finite set S in $\mathbb{R}^3$ has a unique point C_S (called centroid) such that the sum of distances from C_S to every point in S is minimized.

(d) Let g act on O_x. Then what is the centroid of $g \cdot O_x$?

(e) Use the fact that G acts freely on $\mathbb{R}^3$ to show that g is the identity.

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[3] Hint: Show that it has a nontrivial elements of finite order.

3 Banach-Tarski Paradox

In 1924, Banach and Tarski proved that a solid ball in $\mathbb{R}^3$ can be decomposed into a finite number of pieces, which can then be reassembled to form two disjoint balls, each of the same volume as the single ball you started with.

Question 3.1. What is an anagram of the word "Banach-Tarski"?

Before we decompose our unit ball in $\mathbb{R}^3$, consider following "modest paradox".

Example 3.2. Given a broken circle $S^1 \setminus \{\text{point on the circle}\}$, we can break it into finite pieces and reassemble them back into a circle S. Let $S^1 \subseteq \mathbb{C}$ be the unit circle

$$S^1 := \{x : |x| = 1\}.$$

We can let the point that we are deleting be $1 = e^{i(0)}$. Define $A = \{e^{in} : n \in \mathbb{N}\}$ and $B = S^1 \setminus (\{e^{i(0)}\} \cup A)$ be all other points on S^1. All the e^{in} points are distrinct for different $n \in \mathbb{N}$ because of the irrationality of 2π. We map every $e^{in} \in A$ to $e^{i(n-1)}$ via multiplying on the left by e^{-i} while fixing B. This rotation is an isometry in $\mathbb{C}$. Then we have completed the circle by combining the rotated A and B.

This is an example of breaking a thing into finitely many pieces and resemble into the original thing. We have a name for such phenomenon.

Definition 3.3. Let G acts on a set X, and suppose $U, V \subseteq X$. We say that U and V are G-**equidecomposable** if U and V can be partitioned into $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ respectively such that for every i, there exists some $g_i \in G$ such that $g_i \cdot A_i = B_i$.

In the Example 3.2, $S^1 \setminus \{\text{point on the circle}\}$ is U, and S^{-1} is V. We have decomposed $S^1 \setminus \{\text{point on the circle}\}$ into A, B. By rotating A using the rotation map e^{-i}, and applying the identity rotation to B, we obtain A' and B' that union to give a decomposition V. Hence $S^1 \setminus \{\text{point on the circle}\}$ and S^1 are equidecomposable.

Let us start with the surface S of the unit solid ball B in $\mathbb{R}^3$.

Lemma 3.4. *The group of rotations in $\mathbb{R}^3$ contains a subgroup that is isomorphic to F_2, denoted by F, generated by $\{l, o\}$.*

This lemma gives us a free group F and we can show that F_2 (a tree!!!) is equidecomposable to two copies of itself. We can start by saying that we can let $W(l)$ be the set of all words that start with l and define $W(o), W(l^{-1}), W(o^{-1})$ similarly. Then we can partition the tree into four parts:

$$W = \{e\} \cup W(o) \cup W(o^{-1}) \cup W(l) \cup W(l^{-1}).$$

Question 3.5. What is $lW(l^{-1})$?

Therefore, $W(o) \cup W(o^{-1}) \cup W(l^{-1})$ can be written as

$$lW(l^{-1}).$$

Hence

$$W = lW(l^{-1}) \cup W(l) = oW(o^{-1}) \cup W(o).$$

Using this, we can device a paradoxical decomposition of our tree, or the Cayley graph, $\Gamma(F)$. Consider the following. Define $B = o^{-1} \cup o^{-2} \cdots$.

$$\begin{aligned} A_1 &= W(o) \cup \varnothing \\ A_2 &= W(o^{-1}) \setminus B \\ A_3 &= W(l) \\ A_4 &= W(l^{-1}). \end{aligned}$$

Now the magic happens:

$$\begin{aligned} oA_2 &= A_2 \cup A_3 \cup A_4 \\ lA_4 &= A_1 \cup A_4 \cup A_2 \cup B \\ A_1 &= A_1 \\ A_3 &= A_3 \\ B &= B. \end{aligned}$$

Therefore, when F acts on the sphere, we decompose S into two copies of itself as well! To make this precise, for every point x, F takes x to a set of points on the surface. This is the orbit of x under F, denoted by O_x.

Question 3.6. For two points $p, q \in S$, if $p \in O_q$, what is O_p?

You can check that the relation $p \sim q$ if $O_p = O_q$ is an equivalence relation. Each O_p for some orbit representative p is a equivalence class. The orbits, or the equivalence classes, form a partition S. By the Axiom of Choice, we can pick one orbit representative from each orbit, and call the set of all orbit representatives D.

We think of D as our home depot, and every point y on S is reachable under F from some point x in D . The set of destinations that x travels to is a copy of F. Since we have shown that $\Gamma(F)$ is equidecomposable to two copies of itself, we just showed that the whole sphere can also be equidecomposable to two copies itself.

But wait a second....All of these relies on the assumption that we have such a free subgroup of the rotation group in $\mathbb{R}^3$. How do we find such a group to start with?

We define our group to be generated by a rotation l around z-axis by angle $\arccos(\frac{1}{3})$ and a rotation o around x-axis by angle $\arccos(\frac{1}{3})$. That is,

$$l = \begin{pmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

In homework, you will see that these two rotations indeed generated a free group.

Furthermore, we need to show that the argument above can be extended from the surface to the whole solid unit ball. Let B be the unit solid ball in $\mathbb{R}^3$.

Lemma 3.7. *B is equidecomposable with $B \setminus \{(0,0,0)\}$.*

3.1 Exercises for Day 4

Exercise 3.8. Show that the two rotations l, o indeed generates a free subgroup of the rotation group in $\mathbb{R}^3$.

Exercise 3.9. Let B be the unit solid ball in $\mathbb{R}^3$. Show that B is equidecomposable with $B \setminus \{(0,0,0)\}$.

4 Ping Pong Lemma for Two Players

Groups, like
men, are judged
by their actions.

Anonymous

In this class, we have seen that free group are really nice and have cool properties. But where can we find them? This section is mainly following [Mei08]. Consider the following example.

Definition 4.1. A **homeomorphism** of $\mathbb{R}$ is a continuous, incjective, surjective map $\phi : \mathbb{R} \to \mathbb{R}$, whose inverse is also continuous.

Example 4.2.

- $f(x) = x^3$;
- $f(x) = 7 - x$;
- $f(x) = x + \sin(x)$.

Question 4.3. The collection of homeomorphisms of $\mathbb{R}$ forms a super large group denoted by $\mathrm{Homeo}(\mathbb{R})$. The group operation is function composition. What is the identity of the group?

Many interesting groups are subgroups of $\mathrm{Homeo}(\mathbb{R})$. Our goal today is to find a free subgroup of $\mathrm{Homeo}(\mathbb{R})$.

Example 4.4. Consider the subgroup H generated by $f(x) = x^3$ nad $g(x) = x^5$. It is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ by the following map $\phi : H \to \mathbb{Z} \oplus \mathbb{Z}$:

$$\phi : (x^{3^m})(x^{5^n}) \mapsto (m, n).$$

We can see that this is indeed an isomorphism by checking injectivity and surjectivity.

Definition 4.5. A **free abelian group** of rank r is isomorphic to $\mathbb{Z}^r$. It is an abelian group with a basis, or an linearly independent generating set B where B has r elements.

Our goal is to construct explictly an example of free subgroup of $\mathrm{Homeo}(\mathbb{R})$.

Example 4.6. Let

$$h(x) = \begin{cases} 4x & x \in [0, \frac{1}{5}] \\ \frac{4}{5} + \frac{1}{4}(x - \frac{1}{5}) & x \in [\frac{1}{5}, 1] \end{cases}$$

and extend h to the real line by defining

$$f(x) = \lfloor x \rfloor + h(x - \lfloor x \rfloor)$$

and

$$g(x) = f(x - \frac{1}{2}) + \frac{1}{2},$$

so g is a shifted copy of f.

Let

$$X_f = \cup_{n \in \mathbb{Z}} \left[n - \frac{1}{5}, n + \frac{1}{5} \right]$$

and

$$X_g = \cup_{n \in \mathbb{Z}} \left[i + \frac{1}{2} - \frac{1}{5}, i + \frac{1}{2} + \frac{1}{5} \right].$$

Notice that

$$f\left[\left(n + \frac{1}{5}, n + \frac{4}{5} \right) \right] \subseteq \left(n + \frac{4}{5}, n + 1 \right) \subseteq X_f.$$

We can use inductive arguments to show that if $x \notin X_f$ then $f^k(x) \in X_f$ for any positive integer k. Similarly, this holds for g.

We show that f, g generate a free group. Notice that $X_f \cap X_g = \varnothing$ and $\frac{1}{4} \notin X_f \cup X_g$. Let w be any word in $W(f, g)$.

$$w(\frac{1}{4}) = w_1^{k_1} \cdots w_m^{k_m} (\frac{1}{4}) \in X_f \cup X_g.$$

Since $\frac{1}{4} \notin X_f \cup X_g$, so $w(\frac{1}{4}) \neq \frac{1}{4}$ and $w \neq e$.

Lemma 4.7 (Ping Pong Lemma for Two Players)**.** *Suppose a and b generate a group G that acts on a set X. If*

(a) X has a disjoint nonempty subsets X_a and X_b;

(b) $a^k X_b \subseteq X_a$ and $b^k X_a \subseteq X_b$ for all $k \neq 0$,

then G is isomorphic to a free group of rank 2.

Question 4.8. Can you draw pictures of how a^k and b^k for nonzero k act on X?

Proof. Let $g = a^* b^* a^* \cdots b^* a^*$ in G, where all the $*$'s are non-negative integers. By (b), $g \cdot X_b \subseteq X_a$. Hence $g = e_G$.

Now we can show that any element is conjugate of this form.

Definition 4.9. Let $g \in G$. The conjugate of g by h is hgh^{-1}.

Let's prove this. Say we have the element $g = a^{42} b^6 a^{163} b^{47}$. Then we can always have

$$g = a^1 (a^{42} b^6 a^{163} b^{47} a^1) a^{-1}.$$

This illustrates how you can always write an element as a conjugate of the form. Since g is the conjugate of an nontrivial element, and conjugate of an nontrivial element is nontrivial, g must is nontrivial. □

Example 4.10. Let $\mathrm{GL}_2(\mathbb{Z})$ be the group of 2×2 matrices with determinant equal to ± 1. We can identify some free subgroups of $\mathrm{GL}_2(\mathbb{Z})$ by inspecting its action on $\mathbb{R}^2$.

For any integer $m \geq 2$, the matrices

$$l = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ and } o = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

generate a free subgroup of rank 2. To show this we will use Ping Pong Lemma for Two Players.

First, we can show that for all $k \neq 0$,

$$l^k = \begin{pmatrix} 1 & km \\ 0 & 1 \end{pmatrix} \text{ and } o^k = \begin{pmatrix} 1 & 0 \\ km & 1 \end{pmatrix}.$$

Consider the following disjoint sets in $\mathbb{R}^2$,

$$X_l = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| > |y| \right\} \text{ and } X_o = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| < |y| \right\}$$

We can show that $l^k(X_o) \subseteq X_l$ and $o^k(X_l) \subseteq X_o$ and use Ping Pong Lemma for Two Players to conclude that $\langle l, o \rangle$ is a free group.

References

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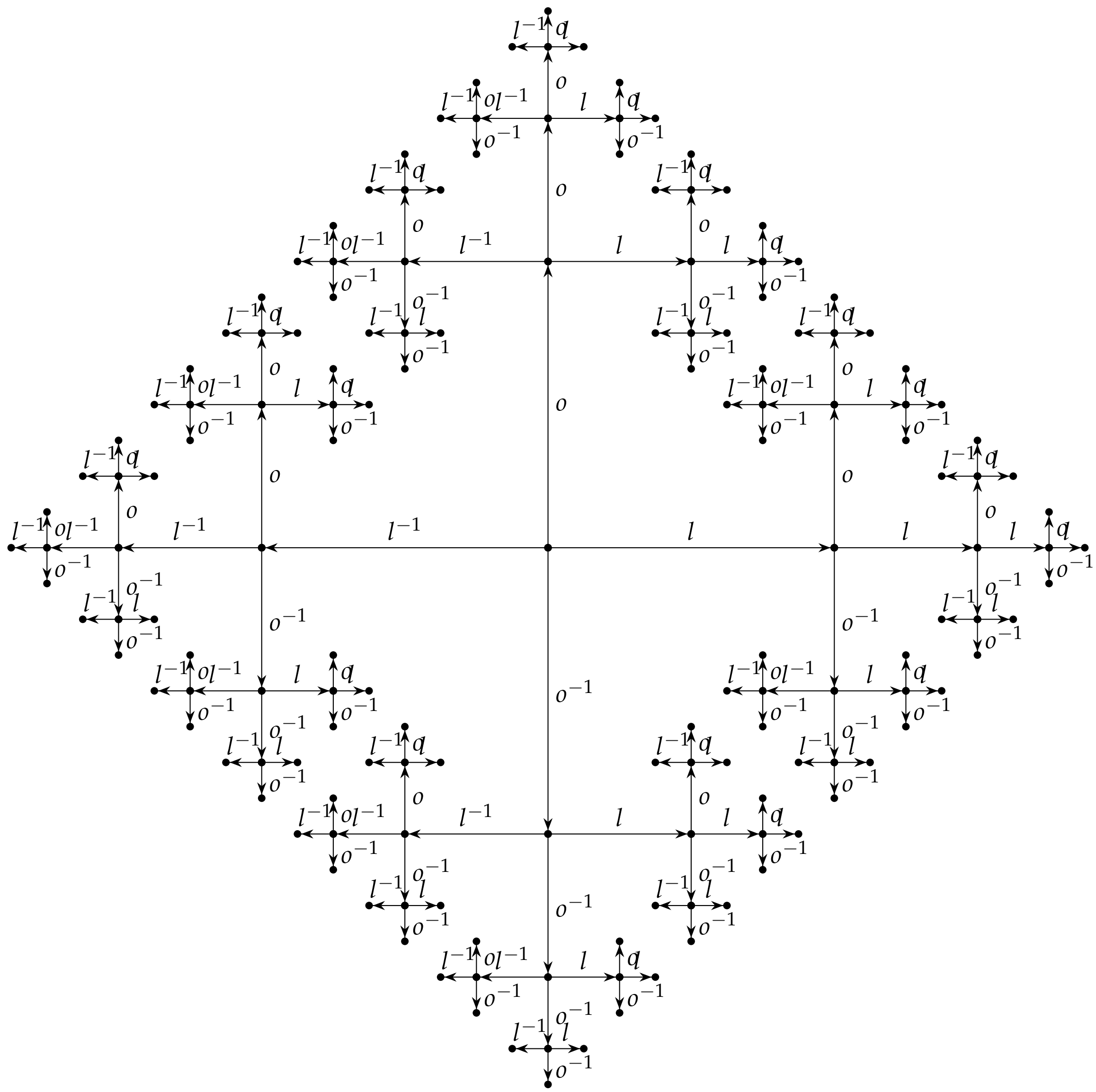

Figure 1: A graph Γ that explores all possibilities of words in our "lol"-language. In other words, this is a Cayley graph of F_2 with basis $\{l, o\}$.