EQUIVARIANT LOG-CONCAVITY OF INDEPENDENCE SEQUENCES OF CLAW-FREE GRAPHS

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Abstract. We show that the graded vector space spanned by independent vertex sets of any claw-free graph is strongly equivariantly log-concave, viewed as a graded permutation representation of the graph automorphism group. Our proof reduces the problem to the equivariant hard Lefschetz theorem on the cohomology of a product of projective lines. Both the result and the proof generalize our previous result on graph matchings. This also gives a strengthening and a new proof of results of Hamidoune, and Chudnovsky–Seymour.

1. Introduction

A graph G is claw-free if no induced subgraph is the bipartite graph $K_{1,3}$. An independent set of a graph G is a set of nonadjacent vertices. The independence sequence of a claw-free graph is *log-concave*: for all $1 \leq k \leq \ell$, the numbers I_j of independent sets of size j satisfies that

$$I_{k-1}I_{\ell+1} \leq I_k I_\ell.$$

First, Hamidoune [Ham90] gave a combinatorial proof of a slightly stronger result than mere log-concavity. Then, Chudnovsky and Seymour [CS07] proved it by showing that the generating polynomial has only real roots, which is well-known to impliy log-concavity.

It is often interesting to ask if a certain behavior of a mathematical object respects the underlying symmetry. The notion of equivariant log-concavity was introduced by Gedeon, Proudfoot and Young [GPY17] as a natural categorification of logarithmic concavity. Recently, it is used to study various log-concavity behaviors with respect to a natural group action in the contexts of topology, geometry and combinatorics.

Let Γ be a finite group, a Γ-representation $V_\bullet$ is **strongly equivariantly log-concave** if for all $1 \leq k \leq \ell$,

$$V_{k-1} \otimes V_{\ell+1} \subseteq V_k \otimes V_\ell$$

as a Γ-subrepresentation.

We highlight some known equivariantly log-concave graded representations that are of combinatorial, geometric, and topological interests in the literature:

Theorem 1.1. *(A) The $V_\bullet^n$ given by the q-binomial coefficients for a fixed n as a $GL_n(\mathbb{F}_q)$-representation is strongly equivariantly log-concave* [PXY18, Proposition 6.7]*.*

(B) The rational cohomology $H^*(\mathrm{Conf}(n,\mathbb{C}),\mathbb{Q})$ of the configuration space of n points in $\mathbb{C}$ as an S_n-representation is strongly equivariantly log-concave for degrees ≤ 14 [MMPR21]*.*

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(C) The rational cohomology $H^*(\mathrm{Conf}(n, \mathbb{R}^3), \mathbb{Q})$ *of the configuration space of* n *points in* $\mathbb{C}$ *as an* S_n*-representation is strongly equivariantly log-concave for degrees* ≤ 14 [MMPR21].

(D) The $V_\bullet^n$ *of even degrees of the intersection homology of the complex affine hypertoric variety of the root system of* $\mathfrak{sl}_n$*, viwed as an* S_n*-representation is strongly equivariantly log-concave for degrees* ≤ 14 [MMPR21].

(E) The $V_\bullet^G$ *given by matchings in a graph* G *as an* $\mathrm{Aut}(G)$*-representation is strongly equivariantly log-concave* [Li22].

(F) The $V_\bullet^n$ *given by* k*-subsets in* $[n]$ *as an* S_n*-representation is strongly equivariantly log-concave (as a special case of* [Li22]*).*

The aim of the present paper is to study the equivariant log-concavity of the following graded representation. Let G be a claw-free graph. Let $\mathbb{I}_j$ denote the set of independent vertex sets of size j. The automorphism $\mathrm{Aut}(G)$ naturally acts on all independent vertex sets, and each $\mathbb{I}_j$ is invariant under this action. Define the graded representation of $\mathrm{Aut}(G)$

$$V_\bullet^G = \bigoplus_{j \geq 0, I \in \mathbb{I}_j} \mathbb{C}I,$$

and it admits a grading given by cardinalities.

The primary aim of the paper is to prove the following theorem.

Theorem 1.2. *For any claw-free graph* G*, the graded vector* $V_\bullet^G$ *is strongly equivariantly log-concave.*

Remark 1.3. Our proof uses combinatorics inspired by the work of Kratthenthaler [Kra96] to reduce the problem to the equivariant hard Lefschetz theorem on a Boolean algebra, or the cohomology of a product of projective lines, a generalization of the method in the author's previous work on graph matchings [Li22]. The result specializes to our previous result on graph matchings by taking the line graph $L(G)$ of a graph G: The line graph $L(G)$ of a graph G consists of vertices each for every edge in G and edges each for every common vertex shared by two edges in G. For example, every cycle graph C_n with n edges has its line graph isomorphic to itself, and the line graph of K_4 is the 1-skeleton of the hypersimplex $\Delta(2, 4)$. A matching on G of size k yields an independent vertex set in $L(G)$ of size k. Line graphs are claw-free, by construction, but not all claw-free graphs are line graphs. For example, any complete graph K_n for $n > 3$ cannot be the line graph of a graph, but K_n is claw-free.

Remark 1.4. Taking dimensions immediately covers the previous results of Hamidoune, and Chudnovsky–Seymour, thus giving new proofs to these results.

Remark 1.5. Communicated by Eric Ramos and Nick Proudfoot, the group consisting of Melody Chan, Chris Eur, Dane Miyata, Nick Proudfoot, Eric Ramos, Lorenzo Vecchi, Claudia Yun, was studying if the graded $\mathrm{Aut}(T)$-representation of the independence sequence of a tree T is strongly equivariantly log-concave. They provided a counterexample, the star graph with 6 leaves, to disprove the statement. Note that this counterexample is "claw-ful", quite the opposite of "claw-free". Morally speaking, the enigmatic "claw" structure seems to be an obstruction to the equivariant log-concavity of independence sequence of a tree, but the lack thereof turns out to be crucial in our proof of Theorem 1.2.

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2. Proof of the main theorem

In this section, we prove the main theorem. The main idea is to construct a family of $\mathrm{Aut}(G)$-equivariant injections

$$V_{k-1} \otimes V_{\ell+1} \hookrightarrow V_k \otimes V_\ell$$

for all $1 \leq k \leq \ell$ by reducing to the equivariant hard Lefschetz operator on a Boolean algebra, or the cohomology of a product of projective lines, via the combinatorics of the independent vertex sets. This method is inspired by Krattenthaler's combinatorial proof of the log-concavity of graph matching sequence [Kra96].

Fix a graph G and $1 \leq k \leq \ell$, for each pair I, J in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$, consider the induced subgraph on the symmetric difference of I and J, i.e., $(I \smallsetminus J) \cup (J \smallsetminus I)$, denoted by $G_{I,J}$. The components in $G_{I,J}$ can only be either a path or a cycle, because G is claw-free and I, J are independent vertex sets. Consider all the components in $G_{I,J}$ that are paths of even lengths, i.e., paths that contain odd number of vertices in $I \cup J$, denoted as $C_{I,J}$. ("C" for "chains".) Color vertices from I with blue, and J with pink. Note that each path in $C_{I,J}$ has both endpoints color blue or pink. Now we do some counting: Let $P_{I,J}$ resp. $B_{I,J}$ be the number of pink resp. blue paths in $C_{I,J}$.

(a) $P_{I,J} + B_{I,J} = |C_{I,J}|$;
(b) $P_{I,J} - B_{I,J} = (\ell + 1) - (k - 1) \geq 2$.

From these, we know

$$2B_{I,J} \leq B_{I,J} + P_{I,J} - 2 = |C_{I,J}| - 2, \text{ and therefore, } B_{I,J} \leq \frac{|C_{I,J}|}{2} - 1. \tag{1}$$

Our next step is to decompose each of $V_{k-1} \otimes V_{\ell+1}$ and $V_k \otimes V_{\ell+1}$ into a direct sum of Boolean algebras on certain partitions in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ and $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$.

Definition 2.1. Two pairs $(I, J), (I', J')$ of independent vertex sets are equivalent if $I \cup J = I' \cup J'$ and I resp. J agrees with I' resp. J' outside of $C_{I,J}$ or $C_{I',J'}$.

One verifies using arguments in [Li22] that this indeed gives partitions $\Pi_{k-1,\ell+1}$ and $\Pi_{k,\ell}$ on $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ and $\mathbb{I}_k \times \mathbb{I}_\ell$ respectively.

For any part $P \in \Pi_{k-1,\ell+1}$ and each pair (I, J) in P, we associate a set of pairs of independent vertex sets in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ as follows. For each path in $C_{I,J}$ with endpoints colored blue, we swap the colors on all the vertices in this path from pink to blue and from blue to pink. This swapping produces a path in $C_{I,J}$ with endpoints color pink. Now collect all the blue resp. pink vertices in $G_{I,J}$ and record that as I' resp. J'. Since I' resp. J' now has k resp. ℓ vertices, the pair (I', J') is in $\mathbb{I}_k \times \mathbb{I}_\ell$. Repeat for every path in $C_{I,J}$, we obtain a subset $N_{I,J}$ in $\mathbb{I}_k \times \mathbb{I}_\ell$. Using a similar argument as in [Li22, Section 2.2], we verify that $N_{I,J}$ is a part of $\Pi_{k,\ell}$, denoted as P'. Now define a map

$$\Phi_{k,\ell}\colon V_{k-1}\otimes V_{\ell+1}\to V_k\otimes V_\ell,\quad I\otimes J\mapsto \frac{1}{|N_{I,J}|}\sum_{(I',J')\in N_{I,J}} I'\otimes J'.$$

Using similar arugment as in [Li22, Section 2.2], one verifies that $\Phi_{k,\ell}$ is $\operatorname{Aut}(G)$-equivariant.

To show injectivitity, we consider the following vector space for any part P in $\Pi_{k-1,\ell+1}$

$$V_{k-1,\ell+1}(P) := \operatorname{Span}_{\mathbb{F}}\{I\otimes J \mid (I,J)\in P\}.$$

We now realize $V_{k-1,\ell+1}(P)$ as a categorification of the B_Pth level of the Boolean lattice on C_P. Consider the map

$$\beta_P\colon P\to \binom{C_P}{B_P},\quad (I,J)\mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is well-defined by the construction of paths of blue endpoints in C_P and bijective using a similar argument in [Li22]. Next, we consider the vector space

$$V_{C_P,B_P} := \operatorname{Span}_{\mathbb{F}}\left\{B \mid B\in \binom{C_P}{B_P}\right\},$$

and define

$$\underline{\beta_P}\colon V_{k-1,\ell+1}(P)\to V_{C_P,B_P},\quad I\otimes J\mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is an isomorphism of vector spaces, because β_P is a bijection on the bases.

Then, we do the same procedure for $\mathbb{I}_k\times\mathbb{I}_\ell$. We define vector spaces $V_{k,\ell}(P')$, $V_{C_P,B_{P'}}$ and the maps $\beta_{P'}$ and $\underline{\beta_{P'}}$ similar to those for P. Note that, by construction,

$$B_{P'} = B_P + 1 \quad\text{and}\quad C_{P'} = C_P.$$

Finally, for each P in $\Pi_{k-1,\ell+1}$, define the linear map

$$L_P\colon V_{C_P,B_P}\to V_{C_{P'},B_P+1},\quad B\mapsto \frac{1}{|C_P|-B_P}\sum_{B\subseteq B'\in\binom{C_P}{B_P+1}} B'.$$

Crucially, L_P is a hard Leftschetz operator on the Boolean algebra spanned by all subsets of C_P, where the grading is given by cardinality. It is injective for degrees $B_P\le |C_P|/2-1$. This operator and its injectivity on the lower half graded pieces have been studied in various contexts. We invite the reader to see proofs of various flavors: [Sta80], [Sta83, The hard Lefschetz theorem], [HW08, Proposition 7], [HMM+13], [Sta13, Theorem 4.7] and [BHM+20, Theorem 1.1(3)].

By construction, the following diagram commutes:

$$\begin{array}{ccc} V_{k-1,\ell+1}(P) & \xrightarrow[\cong]{\beta_P} & V_{C_P,B_P} \\ \downarrow{\scriptstyle \Phi_{k,\ell}} & & \downarrow{\scriptstyle L_P} \\ V_{k,\ell}(P') & \xrightarrow[\cong]{\beta_{P'}} & V_{C_{P'},B_P+1}. \end{array}$$

Therefore, $\Phi_{k,\ell}$ is injective from $V_{k-1,\ell+1}(P)$ to $V_{k,\ell}(P')$.

Note that by construction,

$$V_{k-1} \otimes V_{\ell+1} = \bigoplus_{P \in \Pi_{k-1,\ell+1}} V_{k-1,\ell+1}(P) \cong \bigoplus_{P \in \Pi_{k-1,\ell+1}} V_{C_P,B_P}.$$

Then the last sentence of the previous paragraph implies that $\Phi_{k,\ell}$ is injective on $V_{k-1} \otimes V_{\ell+1}$.

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